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Hybrid k_T-factorization at next to leading order

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Kraków, 2020 - 2024

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Acknowledgements

I would like to emphasise that my Ph.D. studies would have not happen without my supervisor Andreas van Hameren. His intelligence, knowledge and excellent mentor skills allowed me to participate as well as help in such a big project and finally finish this thesis. I cannot find words that can express my gratitude.

Also, I would like to thank our collaborators who work or worked in the Institute, i.e. Krzysztof Kutak, Aleksander Kusina, Sebastian Sapeta, Maciej Skrzypek, Zbigniew Wąs, Krzysztof Golec-Biernat, Etienne Blanco, Martin Rohrmoser, Alessandro Giachino, Souvik Adhya, Rene Poncelet, Richard Ruiz, Peter Baron, Nasim Derakhshanian. Sometimes it was one word or a short conversation, and sometimes it was more detailed advice – for all of this I thank you a lot.

Last but not least, I would like to thank my dear wife, as her help and faith in me was an indispensable fuel for me. I am also thankful to all my friends and family for their permanent support during my Ph.D. studies.

Abstract

In this thesis, we employ k_T-factorization to study quantum chromodynamics (QCD) in the high-energy limit. The k_T-factorization differs from collinear factorization by incorporating the transverse momentum component carried by one initial-state parton.

The main goal of this study is to extend the hybrid factorization formula to next-to-leading order (NLO). We establish both real and virtual contributions. We identified all divergences present in obtained formulas. Some divergent terms cancel straightforwardly, while others remain and can partly be attributed to the PDFs (parton distribution function), but not completely and require the interpretation as corrections to target impact factors and evolution kernel. We also provide a general framework for calculating NLO impact factors and reproduce the known formulas for inclusive NLO gluon and quark impact factor corrections.

We also investigated the application of the subtraction method to real radiation divergences at NLO in hybrid k_T-factorization. Unlike the subtraction method in collinear factorization, where momentum recoil is handled differently, here we distribute the recoil over the initial-state partons. We applied numerical methods to evaluate the finite integrals, ensuring their convergence. Additionally, we found that the divergences present in the integrated subtraction terms cancel against those from the virtual contributions.

The thesis is based on the publications [1, 2]. The research conducted during this Ph.D. study forms a significant component of these two publications.

Streszczenie

W tej rozprawie, wykorzystujemy faktoryzację pędu poprzecznego do badania chromodynamiki kwantowej w zakresie wysokich energii. W porównaniu do faktoryzacji współliniowej, faktoryzację pędu poprzecznego charakteryzuje niezerowa prostopadła składowa pędu niesionego przez jeden początkowy parton.

Głównym celem tej pracy jest opracowanie poprawki pierwszego rzędu do wyrażeń hybrydowej faktoryzacji. Opracowaliśmy część rzeczywistą i wirtualną rzeczonej poprawki. Znaleźliśmy i zidentyfikowaliśmy wszystkie obecne rozbieżności pojawiające się we wzorach. Niektóre z nich redukują się bezpośrednio, pozostałe w części można dołączyć do funkcji rozkładu partonów oraz należy interpretować je jako poprawka do zderzeniowego czynniku wpływu i źródła ewolucji. Również stworzyliśmy generalny schemat obliczeń czynników wpływu i odtworzyliśmy istniejące wyrażenia dla poprawek pierwszego rzędu dla kwarkowych i gluonowych czynników wpływu.

Zastosowaliśmy również metodę odejmowania do rozbieżności pochodzącej od części rzeczywistej poprawki dla hybrydowej faktoryzacji pędu poprzecznego. W przeciwieństwie do faktoryzacji współliniowej, gdzie metoda odejmowania traktuje pęd odrzutu w inny sposób, w naszej sytuacji odrzut może być rozprowadzony wśród stanów początkowych partonów. Przy pomocy metod numerycznych obliczyliśmy całki skończone, upewniając się o ich zbieżności. Dodatkowo, spostrzegliśmy, że rozbieżności obecne w czynniku odejmowania, redukują się z rozbieżnościami obecnymi w części wirtualnej poprawki.

Niniejsza rozprawa bazuje na publikacjach [1, 2], dla których badania przeprowadzone w czasie doktoratu są kluczowym składnikiem.

Contents

1 Introduction

Since the 17th, century physicists have been seeking for laws of Nature. Numbers of people have been struggling to work out how the world is organised. There are two ways that can be followed, experimental or theoretical. Recent findings of particle physics research have brought the answer to what the universe consists of and what are the interactions between the littlest blocks called elementary particles. This model of elementary particles and their interactions is known as the Standard Model (SM). Its component, the Quantum Field Theory (QFT), provides an excellent description of subatomic forces. One of the field theories which illustrates the behaviour of quarks and gluons (main building elements of hadrons, for instance proton or neutron) is Quantum Chromodynamics (QCD). Despite having been discovered many decades ago, QCD is considered the most correct theory to describe the strong interaction, one of the four fundamental interactions (the gravity, the electromagnetism, the week interaction, the strong interaction). There is no empirical evidence in disagreement with QCD. Such results can be obtained by applying in calculations the perturbation theory, which demands coupling constant to be small enough. Experimental research in this field relies on colliding elementary particles, hadrons or entire nuclei. The quantity measured or calculated in experiments is termed the cross section and it shows the probability of a given process.

Due to technological improvement, the energy of collisions has increased significantly since the beginning of the 21st century. The crucial point was in 2007 when Large Hadron Collider (LCH) had its first run. Experiments which took place at LHC (and other laboratories) provided enormous amount of data consisting of proton-proton, proton-nucleus, nucleus-nucleus and other collisions. Due to the increase of the hadrons' energy numerous unknown processes were discovered, and thus, the precision of the theory predictions was not sufficient anymore. The needs of QCD require higher computation power then it is currently possible to achieve. In order to avoid such restrictions, some methods of simplification of calculations have been developed.

1.1 Parton model

The structure of a hadron is well described by the so-called parton model. Its main idea posits that any hadron is made of the quarks and gluons called together as partons. For instance a proton is composed of three valence quarks surrounded by sea quarks. This phenomenon is illustrated through the parton distribution function (PDF). In the high-energy regime, we express the momenta of any parton, be it a quark or a gluon, using the momenta of a hadron in the following way

$$k^\mu = xp^\mu + k_T^\mu \tag{1}$$

Here, p^μ represents the four-momentum (which is lightlike, i.e., $p^2 = 0$) of the collided hadron, while x denotes the fraction of the hadron's momentum carried by a parton. Additionally, k_T^μ is the transverse component, satisfying the relation $p \cdot k_T = 0$. If there were only three quarks, they would all carry an equal fraction of the proton's (or any hadron's) momentum. However, experiments yield entirely different results (see figure 1).

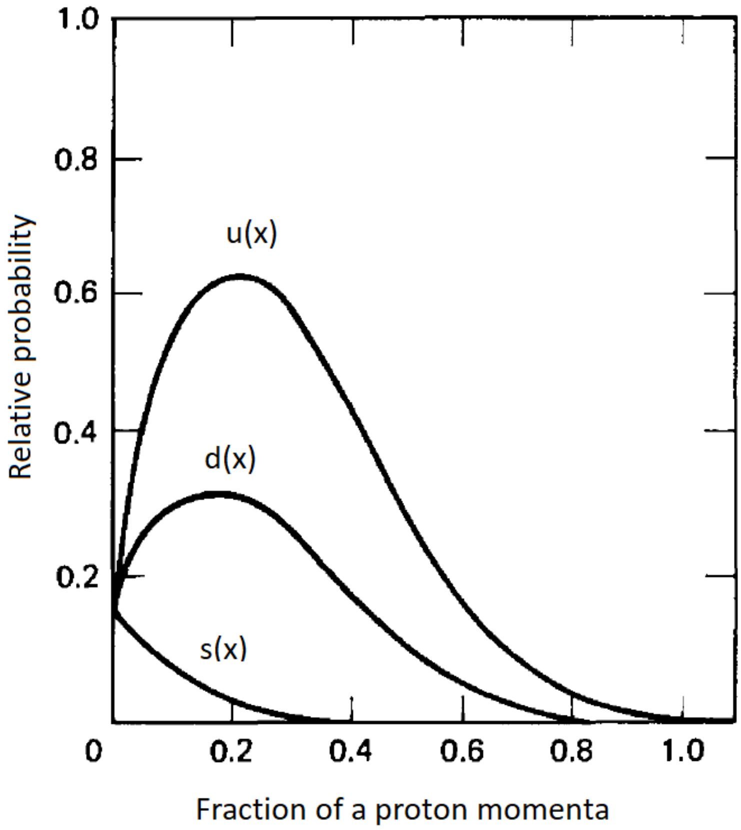


Figure 1: Example of PDFs. Modified from [3].

This picture can only be explained by assuming that a proton is composed of a sea of interacting quarks and gluons. Moreover, pair production and annihilation processes occur, involving other flavors in addition to up and down quarks. We usually represent PDFs as $f_{i/h}(x)$, where the value of this function corresponds to the probability that parton i is involved in a collision and carries a fraction x of the hadron's momentum. As mentioned earlier, PDFs are currently obtained from experiments. There have been attempts to derive them from theory, such as lattice QCD, but satisfactory results are still pending.

The decay function is similar in the final state. It describes the fraction of parton momenta carried by an outgoing hadron. Thus, only a portion of the parton momenta contributes to the final hadron, while the remaining energy is lost. Other properties of the decay function are the same as for the PDF. Both functions are extracted from experiments and are renormalizable, consequently, they contain the singularities of perturbation theory.

The parton model asserts that in any specific collision, regardless of whether it is a strong interaction or otherwise, with any hadron, only one particle from the hadron (be it a quark or gluon) interacts. The rest of the partons act as mere spectators and play no role in the collision. Notably, gluons lack electric charge, so they do not contribute to electromagnetic interactions.

1.2 Collinear factorization

In hadron collisions, such as proton-proton collisions, there exists a relationship between hadronic cross sections and partonic cross sections associated with specific partons. The latter accounts for short-distance interactions in contrast to the long-distance interactions, which I will explain later. The general formula for the differential cross section in the collision of hadrons a and b is expressed as follows

$$d\sigma_{ab} = \sum_{i,j} \int \frac{dx_i}{x_i} \frac{dx_j}{x_j} f_a(x_i) f_b(x_j) d\widehat{\sigma}_{ij}(x_i, x_j) \tag{2}$$

Here, i and j denote two partons, while x_i and x_j represent the fractions of hadron momentum (P_A or P_B). This leads to the definition of the initial parton momenta as follows

$$\begin{aligned} p_i &= x_i \cdot P_A, \\ p_j &= x_j \cdot P_B, \end{aligned} \tag{3}$$

which do not include the transverse component compared to equation (1). The $f_{a/b}$ are two PDFs (parton density functions) describing the respective hadrons (to be explained later), and $d\widehat{\sigma}_{ij}(x_i, x_j)$ represents the partonic cross section describing interactions between partons. This formula is a very simple way to show the factorization. Although we can calculate $d\widehat{\sigma}_{ij}(x_i, x_j)$ using perturbation theory, a straightforward series expansion of Feynman graphs can introduce divergences that complicate the calculations.

To address these divergences, the renormalization framework was developed. The fundamental concept behind this approach is to segregate the infinite portion of the partonic cross section and transfer it to the bare f functions. The convolution of f and the singularities from the partonic cross section yields a new function denoted as $\tilde{f}$. This function is finite and measurable. The validity of this scheme has been verified to any order in perturbation theory, and its independence from the scattering process was demonstrated in [4]. The separation between PDFs and the partonic cross section is the crux of factorization.

Certain subprocesses included in the partonic cross section exhibit infrared (or soft) singularities when the partons are on their mass shell. As demonstrated in [4], the structure of these infrared singularities allows for their factorization from the partonic cross section and their incorporation into the f function. This achievement is thanks to the properties of the Feynman diagrams, where the separation of divergences in the partonic cross section formula are done at the diagram level.

The initially mentioned component of equation (2), represented by f, characterizes the momentum distribution of quarks or gluons. The PDFs are associated with the initial hadron, we also introduce the d function as a decay function. Both PDFs and decay functions are derived from experiments due to their dependence on the long-range nature of the collision process. This aspect of the factorization formula is also referred to as the “soft part.”

Following the work by Ellis, Georgi, and Politzer in 1979 [4], and focusing on the momenta, denoted as P for the initial hadron and P' for the final hadron, we introduce the formulas for differential hadronic cross sections. The remaining momenta are held constant.

$$d\sigma(P) = \sum_{i,h} \int d\tilde{x} \tilde{f}_i^h(\tilde{x}) d\tilde{\sigma}_i^h(\tilde{x}P) \tag{4}$$

$$d\sigma(P') = \sum_{i,h} \int d\tilde{x} \tilde{d}_i^h(\tilde{x}) d\tilde{\sigma}_i^h(P'/\tilde{x}) \tag{5}$$

Here, $\tilde{x}$ differs from the x used in equation (2). This difference arises from the rescaling of parton momentum to facilitate the movement of divergences into the modified PDF ($\tilde{f}$) or decay ($\tilde{d}$) functions. The summations run over all parton types, denoted as i, and helicities, denoted as h. Both equations (4) and (5) are free from infrared singularities.

Factorization enables us to segregate the description of the hadron structure followed by the parton model and the various probabilities governing the ways collisions can occur from the hadron's perspective. Equally important, factorization provides a means to separate and consequently eliminate singularities from the equations.

If one assumes that x is sufficiently large to disregard the transverse momenta k_T of a given parton, we refer to formula (2) as collinear factorization. In this factorization, all partons are considered to be on their mass shell.

If one wishes to specify the final state particles, it becomes necessary to introduce a collision energy scale. All components in the hadronic cross section then depend on this scale. We can restate equation (2) to accommodate a specified final state, denoted as K. By designating the energy scale as Q^2 and writing it explicitly, the equation (2) takes the following form

$$d\sigma_{ab\to K} = \sum_{i,j} \int \frac{dx_i}{x_i} \frac{dx_j}{x_j} f_a(x_i, Q^2) f_b(x_j, Q^2) d\hat{\sigma}_{ij}(x_i, x_j, Q^2). \tag{6}$$

However, some research has demonstrated that certain reactions necessitate the consideration of effects associated with small values of x. In this regime, often referred to as the small-x limit, the k_T part becomes significantly influential. This phenomenon leads to modifications in equation (6), known as k_T-factorization. In this factorization scheme, at least one parton is off-shell. An essential prerequisite for developing the k_T-factorization is a comprehensive understanding of the calculations within collinear factorization, along with the challenges that arise in perturbative corrections. Before we introduce the k_T-factorization formula and the crucial method for obtaining off-shell matrix elements, we will focus on PDFs and collinear factorization at the next-to-leading order (NLO) in the subsequent subsections.

1.3 Virtual and real NLO corrections in collinear factorization

NLO corrections arise from perturbative QCD. The primary quantity involved in the partonic cross section is the amplitude squared, also known as the matrix element. In perturbation theory, the amplitude expression is expanded in terms of the coupling constant, typically denoted as g. However, in cross section calculations, the coupling always appears in even powers. The reason odd powers are not present will be explained later. Therefore, if the cross section at LO (Leading Order) depends on g^4, the NLO contribution is of order g^6. This can be expressed as:

$$d\sigma(g) = d\sigma^{LO}(g^4) + d\sigma^{NLO}(g^6) + d\sigma^{NNLO}(g^8) + d\sigma^{NNNLO}(g^{10}) + \dots \tag{7}$$

Due to the smallness of the coupling constant, higher orders contribute less. However, this is not so apparent at low energy limits in QCD due to the asymptotic freedom of quarks and gluons.

Feynman diagrams are the graphical representation of the expression of the amplitude. A crucial point to note is that there are two conventions for indicating the direction of time flow and the spatial axis in these diagrams. Here, we adhere to the convention where time flow is horizontal (from left to right), and the spatial axis is vertical. An example of such a convention is shown in figure 2.

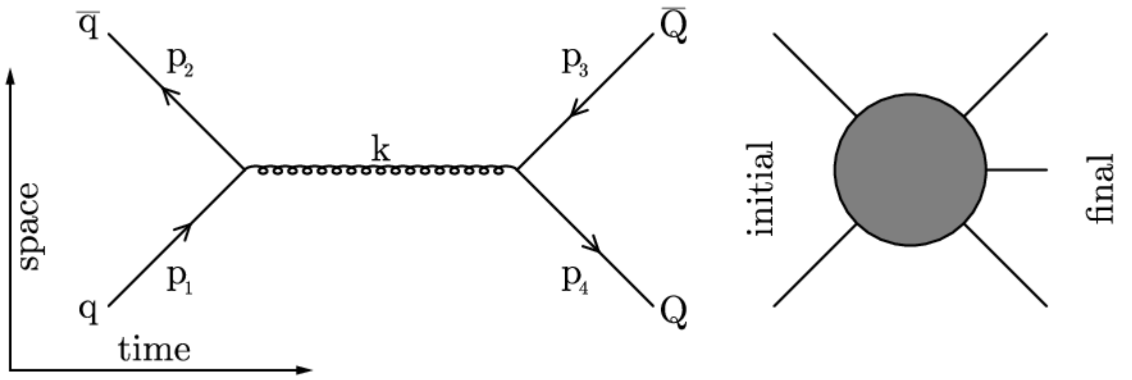


Figure 2: Example of a Feynman diagram.

Quantum field theory dictates that when considering a particular number of initial and final particles, we must account for all possible graphs. Typically, to include all diagrams in one picture, we draw a single blob with external lines (legs) corresponding to incoming and outgoing particles.

The NLO contribution to the cross section consists of two terms: real radiation and virtual correction. The former involves one additional parton in the final state, often referred to as radiative gluon or quark. The latter is the interference between tree-level and one-loop corrections (see figure 3). As mentioned earlier in Equation (7), odd powers of g are absent as they would correspond to the interference between real contributions and the LO or virtual contribution, which lead to inconsistencies in the number of external particles and the phase

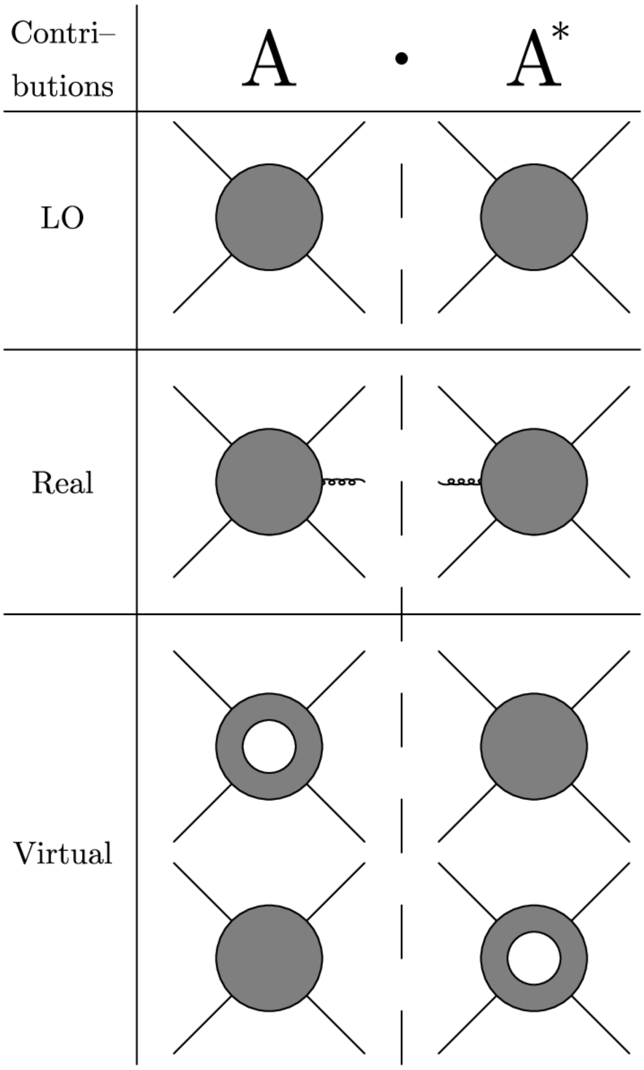


Figure 3: Feynman diagrams of real and virtual contributions at NLO.

space mixing would occur.

The NLO contribution can be schematically written as

$$d\sigma^{NLO} = \int \frac{dx_i}{x_i}\frac{dx_j}{x_j}\Big\{f_a(x_i)f_b(x_j)\big[dV(x_i,x_j) + dR(x_i,x_j)\big]\Big\}. \tag{8}$$

Here, dR and dV correspond to the real and virtual corrections, respectively. The cross section at the lowest-order tree level is denoted by dB, which arises from the typical name for LO, the Born level approximation. Actually this formula is not finite and have to be regularized somehow. Fortunately some divergences cancel and we are left with

$$
\begin{aligned}
d\sigma^{NLO} &= \int \frac{dx_i}{x_i}\frac{dx_j}{x_j}\Big\{ f_a(x_i) f_b(x_j) \Big[dV(x_i,x_j) + dR(x_i,x_j) \Big]_{\text{cancelling}} \\
&+ \Big[f_a(x_i) \frac{-\alpha_s}{2\pi\epsilon} \int_{x_j}^{1} dz_j \mathcal{P}_j(z_j) f_b(x_j/z_j) \\
&\quad + f_b(x_j) \frac{-\alpha_s}{2\pi\epsilon} \int_{x_i}^{1} dz_i \mathcal{P}_i(z_i) f_a(x_i/z_i) \Big] dB(x_i,x_j) \Big\}.
\end{aligned} \tag{9}
$$

In that formula the non-cancelling divergences are written explicitly ($1/\epsilon$). It is important to note that to obtain the complete NLO correction for the hadronic cross section, one must also include NLO corrections to the PDFs, as they also depend on the coupling constant. The divergences from partonic cross section are absorbed into $f^{(1)}$ functions, which are the corrections to the PDFs.

$$
\begin{aligned}
f_b^{(1)}(x_j) - \frac{1}{\epsilon}\int_{x_j}^{1} dz_j \mathcal{P}_j(z_j) f_b(x_j/z_j) &= \text{finite} \\
f_a^{(1)}(x_i) - \frac{1}{\epsilon}\int_{x_i}^{1} dz_i \mathcal{P}_i(z_i) f_a(x_i/z_i) &= \text{finite}.
\end{aligned} \tag{10}
$$

Then we write the complete collinear factorization at NLO

$$
\begin{aligned}
d\sigma^{NLO} &= \int \frac{dx_i}{x_i}\frac{dx_j}{x_j}\Big\{ f_a(x_i) f_b(x_j) \Big[dV(x_i,x_j) + dR(x_i,x_j) \Big]_{\text{cancelling}} \\
&+ \Big[f_a(x_i) \frac{-\alpha_s}{2\pi\epsilon} \int_{x_j}^{1} dz_j \mathcal{P}_j(z_j) f_b(x_j/z_j) \\
&\quad + f_b(x_j) \frac{-\alpha_s}{2\pi\epsilon} \int_{x_i}^{1} dz_i \mathcal{P}_i(z_i) f_a(x_i/z_i) \Big] dB(x_i,x_j) \\
&+ \Big[f_a^{(1)}(x_i) f_b(x_j) + f_a(x_i) f_b^{(1)}(x_j) \Big] \frac{\alpha_s}{2\pi} dB(x_i,x_j) \Big\},
\end{aligned} \tag{11}
$$

which is finite. There are already developed methods for handling divergences. In our work, we will focus specifically on the subtraction method, which we are going to explain in the following subsection.

1.4 Subtraction method

We follow the work of Gábor Somogyi and Zoltán Trócsányi, where they developed a subtraction scheme for QCD jets [5, 6]. In that work, the method is developed at NLO accuracy. To briefly outline the general idea of the subtraction method, let us write equation for partonic cross section, in a manner similar to equation (6), incorporating the definition of initial state momenta as given by (3).

$$
\begin{aligned}
d\sigma^{NLO}_{ab\to n-jets} = \textstyle\sum_{i,j} \displaystyle\int \frac{dx_i}{x_i}\frac{dx_j}{x_j}\Big\{ f_a(p_i,Q^2) f_b(p_j,Q^2) \\
\Big[\textstyle\int_{n+1} d\sigma^R_{ij}(p_i,p_j) J_{n+1} + \int_n d\sigma^V_{ij}(p_i,p_j) J_n + \int_n d\sigma^C_{ij}(p_i,p_j,Q^2) J_n \Big] \Big\}.
\end{aligned} \tag{12}
$$

In this formula, we assume there are n jets in the final state. Thus, the integrals run over all n final-state partons, and the phase space is defined by the jet function J_n. The tree components of the NLO cross section include real radiation, virtual contributions, and the collinear counter term. Since the real contribution involves one additional parton, the phase space has $n+1$ final states, as indicated by the subscript of the jet function and the integral.

In $d = 4$ dimensions, all of these terms are divergent. The typical approach to address this issue is to generalize the number of dimensions as $d = 4 - 2\varepsilon$, perform the calculations, and then take the limit as $\varepsilon \to 0$. This way, all divergences are expressed as poles $1/\varepsilon$. They come from the singular behaviour of the matrix element in two types of regions of the phase space giving soft or collinear singularity. These come from the regions where radiative gluon is soft, and/or parallel to any other parton respectively.

The next step in addressing these singularities involves constructing an appropriate subtraction term, denoted by $d\sigma_{ij}^{R,A}(p_i, p_j)$, to eliminate divergences arising from $d\sigma_{ij}^{R}(p_i, p_j)$ in the d-dimensional phase space within the one-parton unresolved regions. The subtraction term must also be integrable in that region. Subsequently, we subtract $d\sigma_{ij}^{R,A}(p_i, p_j)$ from the real radiation component and reintroduce it to the remaining contributions.

Thus, we write

$$
\begin{aligned}
d\sigma^{NLO}_{ab\to n-jets} = \textstyle\sum_{i,j} \displaystyle\int \frac{dx_i}{x_i}\frac{dx_j}{x_j} \Big\{ f_a(p_i, Q^2) f_b(p_j, Q^2) \\
\Big[\textstyle\int_{n+1} \left(d\sigma_{ij}^{R}(p_i, p_j) J_{n+1} - d\sigma_{ij}^{R,A}(p_i, p_j) J_n \right) \\
+ \textstyle\int_{n} \left(d\sigma_{ij}^{V}(p_i, p_j) + d\sigma_{ij}^{C}(p_i, p_j, Q^2) + \int_1 d\sigma_{ij}^{R,A}(p_i, p_j) \right) J_n \Big] \Big\}.
\end{aligned}
\tag{13}
$$

The first angular bracket is finite in four dimensions by construction. In the second term, the jet function ensures infrared safety, and thanks to the Kinoshita-Lee-Nauenberg theorem, the combination of all terms integrated over n partons remains finite. Our NLO calculations conclude with the summation of two finite integrals, which can be computed using known analytical or numerical methods.

There is a necessity to put the remark that any component in the equation (13) could in general by the sum of terms and appropriate jets function depend on different set of momenta.

In order to illustrate how subtraction method approach works, let us consider a simple example. In the formula (13), we encounter integrals that become divergent due to terms like $1/x$ multiplied by a function dependent on x. For instance, the integral

$$
\int_0^1 dx \frac{1}{x} f(x), \tag{14}
$$

where f is an arbitrary function, is straightforwardly divergent. We make a slight modification by multiplying f by x^{ϵ}, assuming ϵ is very small. In QCD computations, this adjustment is typically performed using dimensional regularization. Then, we write

$$\int_0^1 dx \frac{x^\epsilon}{x} f(x). \tag{15}$$

If we subtract $f(0)$ and then add it back, we end up with two terms: one is finite, and the other contains the singularity, but it is much simpler than the original integral. To demonstrate this, we can apply the Taylor expansion (to be precise, the existence of the first derivative is sufficient for this method to work)

$$\begin{aligned}
\int_0^1 dx \frac{x^\epsilon}{x} f(x) &= \int_0^1 dx \frac{x^\epsilon}{x} \Big[f(x) - f(0)\Big] + \int_0^1 dx \frac{x^\epsilon}{x} f(0) \\
\int_0^1 dx \frac{x^\epsilon}{x} f(x) &= \int_0^1 dx \frac{x^\epsilon}{x} \Big[f(0) + x \cdot f'(0) + \frac{1}{2} x^2 \cdot f''(0) + \ldots - f(0)\Big] + \int_0^1 dx \frac{x^\epsilon}{x} f(0)
\end{aligned} \tag{16}$$

Doing the second integral on the right hand side we get

$$\int_0^1 dx \frac{x^\epsilon}{x} f(x) = \int_0^1 dx \cdot x^\epsilon \left[\cdot f'(0) + \frac{1}{2} x \cdot f''(0) + \ldots\right] + \frac{1}{\epsilon} f(0) x^\epsilon \Big|_0^1 \tag{17}$$

One can easily observe that the first term is finite, as the singular term has been subtracted. This integral can be calculated using numerical methods. On the other hand, the second component is divergent, exhibiting a pole of ϵ, but it is no longer under the integral.

1.5 Auxiliary parton method

To investigate k_T-factorization [7, 8], we need to compute the squared matrix element for an off-shell parton. However, the gauge invariance, which dictates Ward identities to be satisfied by scattering amplitudes and the behavior of gluon propagators, holds only for on-shell external particles. Thus, the pure Feynman rules cannot be directly applied to calculate the off-shell amplitudes and the matrix element. To address this issue, the Lipatov's effective action method [9, 10] was developed. Other methods documented in the literature include those described in [11, 12, 13], which at tree-level, have been shown to lead to results identical to those obtained with the effective action. For this work, the chosen method is the auxiliary parton method, which has also been demonstrated to be applicable at the one-loop level, as shown in [14].

The auxiliary parton method has been developed and elaborated upon [11, 15, 16]. This method enables us to derive the helicity amplitudes and then the squared matrix element for any n parton off-shell process from the $n+1$ parton on-shell case.

To accomplish this, we choose two partons called auxiliary partons, which replace the off-shell gluon. Then, we parametrize the momenta of two auxiliary partons from the on-shell process, using a parameter Λ in such a way that their sum, $k_1^\mu + k_2^\mu$, equals $k^\mu = xp^\mu + k_T$ (see eq. (1)), and both of them have light-like momenta ($k_1^2 = k_2^2 = 0$). To meet these requirements, we express the momenta k_1^μ and k_2^μ as

$$\begin{aligned} k_1^\mu &= \Lambda p^\mu - \frac{\beta^2 k_T^2}{2\Lambda p\cdot q} q^\mu + \beta k_T^\mu, \\ k_2^\mu &= k^\mu - k_1^\mu, \end{aligned} \tag{18}$$

where q^μ is auxiliary light-like momentum, which satisfies

$$q \cdot k_T = p \cdot k_T = 0 \tag{19}$$

and β is defined as

$$\beta = \frac{1}{1+\sqrt{1-1/\Lambda}}. \tag{20}$$

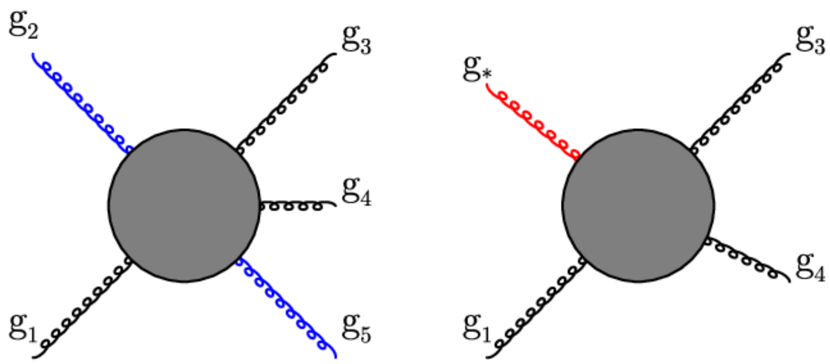


Figure 4: Auxiliary parton method, example on gluons.

The next step is to take the limit as $\Lambda \to \infty$ in the on-shell matrix element squared with auxiliary momenta parametrized by (18). To obtain the correct expression for the squared amplitude with an off-shell parton, we need to include an overall factor based on the on-shell matrix element squared

$$\frac{x^2|k_T|^2}{g_s^2 C_{\mathrm{aux}}\Lambda^2}|M^{\mathrm{aux}}|^2(k_1,k_2,\{p_i\}_{i=1}^n) \xrightarrow{\Lambda\to\infty} |M^*|^2(k,\{p_i\}_{i=1}^n). \tag{21}$$

It explicitly indicates the dependence of the matrix element on the momenta of the final-state partons, denoted by $\{p_i\}_{i=1}^n$. The factor $1/g_s^2$ adjusts the power of the coupling constant, $(x|k_T|)^2$ ensures the matching of the matrix element to the k_T-dependent PDF, and also guarantees the correct on-shell limit ($|k_T| \to 0$). Parameter Λ^{-2} is to correct the power 2 of Λ extracted from matrix element squared. The constant C_{aux} accounts for the difference in color representation and its value depends on the type of auxiliary quarks or gluons.

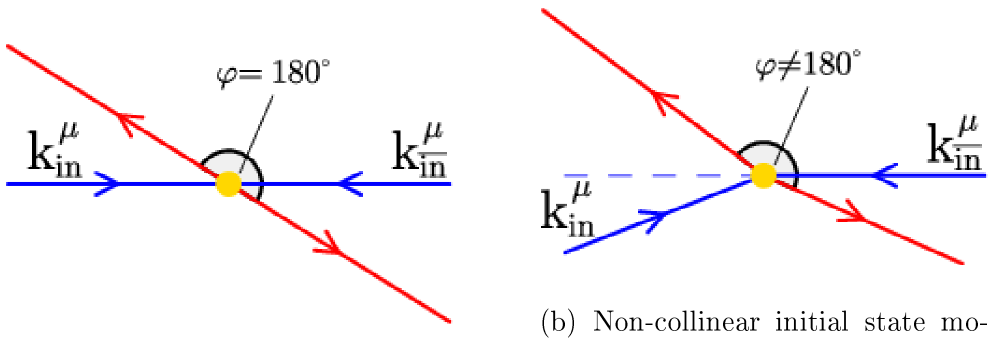
$$C_{\text{aux-q}} = \frac{N_c^2-1}{N_c}, \qquad C_{\text{aux-g}} = 2N_c. \tag{22}$$

Thus it appears that there is a notion of parton-universality that exists up to a trivial color factor. Unfortunately its violation does not stay to a mere factor at NLO as we will see. As in figure 4, we typically choose one initial state and one final state as auxiliary partons. Thus, the off-shell parton is considered as an incoming particle in the desired process.

1.6 k_T-factorization

The study of k_T-factorization represents a natural progression in the development of QCD. This work is motivated both theoretically and from a phenomenological standpoint. The phenomenon of forward jets in physics arises when one of the colliding hadrons contributes a significantly smaller momentum fraction, denoted by x, to the partonic process compared to the other. As a result, forward jets become particularly relevant in the context of k_T factorization. Consequently, all final-state jets are boosted in the direction of the hadron delivering the smaller momentum fraction, as directly observed from equation (1). A more detailed description of forward jets was provided in the work by Michal Deak, Francesco Hautmann, Hannes Jung, and Krzysztof Kutak [17].

Secondly, if one initial state parton has transverse component, there is a momentum imbalance in the final state, which is represented on the figure 5b. To describe such a phenomenon within collinear factorization, at least NLO accuracy is required. Applying k_T-factorization allows for non-trivial distributions of the angle between jets already at LO. One of the main goals of this work is to establish a framework for NLO within k_T-factorization and also to move towards a level of automation at NLO like it exists in collinear factorization [18, 19, 20, 21, 22].



(a) Collinear initial state momenta.

(b) Non-collinear initial state momenta.

Figure 5: Angle between two final jets in dijet production.

The expression for the cross section for a process with an off-shell parton is a generalization of the on-shell case, and it must be consistent with equation (2). It is a guide line that the off-shell cross section reproduces the expression for the on-shell process exactly in the limit as $k_T \to 0$. Therefore, we define the cross section in k_T-factorization as follows

$$d\sigma_{ab} = \sum_{i,j} \int \frac{dx_i}{x_i} \frac{d^2 k_T}{\pi} \frac{dx_j}{x_j} F_i(x_i, k_T) f_j(x_j) dB^*_{ij}(x_i, k_T, x_j). \tag{23}$$

Here, dB^* indicates the partonic cross section at LO with at least one off-shell parton, $F(x_i, k_T)$ is the PDF explicitly dependent on k_T (also called unintegrated parton density function). Rewriting initial state momenta as in equation (3) we have

$$\begin{aligned} p_i^\mu &= x_i \cdot P_A^\mu + k_T^\mu, \\ p_j^\mu &= x_j \cdot P_B^\mu. \end{aligned} \tag{24}$$

As we take the limit $k_T \to 0$ in equation (23), one can easily find that we reproduce equation (2), this requirement for proper definition of cross section within k_T factorization is called the smooth on-shell limit, which for the k_T dependent PDF is $F(x, k_T) \to f(x)\delta(k_T^2)$.

As mentioned in collinear factorization, calculations are now fully automated up to at least 1-loop-level for any process. Presently, our efforts are concentrated on enhancing the precision of theoretical predictions. Developing factorization with k_T involved represents a crucial step in advancing our understanding of QCD. Significant progress has been achieved in this field, resulting in a complete formulation of cross sections in k_T-factorization up to LO. The KaTie Monte Carlo [23] software was developed to automate calculations of cross sections for processes within k_T-factorization at tree-level. The auxiliary parton method is also incorporated into KaTie. Despite further advancements, achieving complete NLO precision remains a challenge, and the computations are not yet fully automated.

During this work, significant progress has been made. We introduced a new framework that allows us to calculate cross sections within the hybrid k_T-factorization at NLO, as described in [1]. In that work, we developed and modified the auxiliary parton method to meet the requirements for NLO calculations. In the following sections, we will provide more detailed explanations regarding this scheme.

Additional progress has been made in addressing singularities present in the real-radiation contribution at NLO. The objective of the work [2] was to apply the subtraction method through the proper construction of the subtraction term. We will elaborate on this work in the section 3.

2 Real radiation

In this section, we will investigate the hybrid k_T-factorization scheme in QCD, where one of the initial state partons carries a non-vanishing transverse components in its momentum. Such a parton is referred to as being space-like or off-shell, while the rest of the partons involved in the process are on-shell. We follow recent work [1], whose main goal was to improve the accuracy of the cross section to NLO. Our objective is to establish an analogous framework as was done for collinear factorization (see section 1.2). We will begin by introducing the real contribution, which originates from the additional parton radiated during the scattering process. In collinear factorization, this contribution exhibits a simple structure with all divergences under control. Some divergences are absorbed into the PDFs' corrections, while the rest cancel out when combining real and virtual corrections.

2.1 Notation

We start with some notation. We write explicitly the Sudakov decomposition for the radiative gluon. Hence, as given in equation (29), we write for the radiative gluon

$$r^\mu = x_r P^\mu + \overline{x}_r \overline{P}^\mu + r_T^\mu. \tag{25}$$

In the phase space with respect to fractions x_r and $\overline{x}_r$ we can distinguish three regions, as it is illustrated in the figure 6. Region A corresponds to the situation where the radiative gluon is collinear to the light-like initial state. The phase space where the radiative gluon is collinear

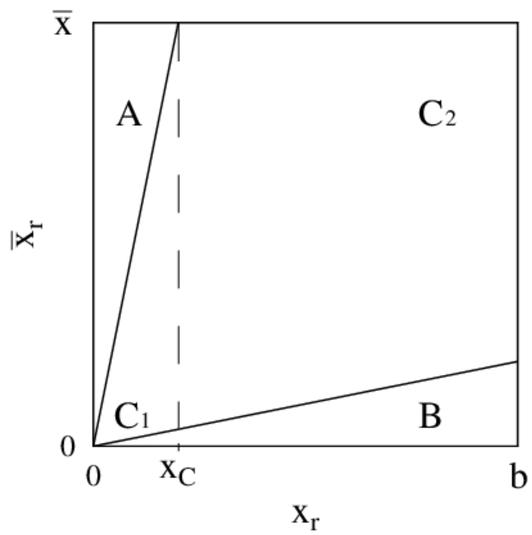


Figure 6: Phase space for radiative gluon.

to the longitudinal component P of the space-like initial state gluon is described by region B. The regions C_1 and C_2 are safe from initial-state collinear singularities.

We assume that the lines separating regions A, B, and C can be expressed as the following functions

$$\overline{x}_r = Cx_r, \tag{26}$$

$$\overline{x}_r = Dx_r, \tag{27}$$

where $C > 1$, $D < 1$ are arbitrary constants or functions.

Then, we rewrite the definition of the cross section within k_T-factorization (23), focusing solely on the partonic perspective and omitting the sum over initial-state partons. Additionally, we will omit the π in the denominator, yielding

$$d\sigma^{(0)} = \int \frac{dx}{x} d^2k_T \frac{d\overline{x}}{\overline{x}} F(x, k_T) f(\overline{x}) dB^\star(x, k_T, \overline{x}). \tag{28}$$

Typically, $F(x_\chi, k_T)$ and $f(\overline{x}_{\overline{\chi}})$ depend on an energy scale, as in equation (6), but here we omit it. While not necessarily customary in the literature, it is worth noting that we refer to PDFs as they are associated with the measures given in equation (28).

To proceed, we define the general decomposition of any four-vector K^μ as follows:

$$K^\mu = x_K P^\mu + \overline{x}_K \overline{P}^\mu + K_\perp^\mu. \tag{29}$$

In that decomposition, the momenta P^μ and $\overline{P}^\mu$ represent the directions of the momenta of the incoming hadrons. We assume these momenta to have positive energy and to satisfy the following conditions

$$P^2 = \overline{P}^2 = 0, \quad 2P \cdot \overline{P} = \nu^2 > 0, \quad P \cdot K_\perp = \overline{P} \cdot K_\perp = 0 \tag{30}$$

with

$$x_K = \frac{\overline{P} \cdot K}{P \cdot \overline{P}}, \quad \overline{x}_K = \frac{P \cdot K}{P \cdot \overline{P}}. \tag{31}$$

In the following, we will often neglect the subscript and write x and $\overline{x}$.

Using decomposition mentioned in equation (29) we define the infinitesimal volume of any light-like momentum

$$d^4K \delta(K^2) = dx_K d\overline{x}_K d^2K_\perp \frac{1}{2x_K} \delta\left(\overline{x}_K - \frac{|K_\perp|^2}{\nu^2 x_K}\right). \tag{32}$$

This notation will also apply to the $K_\perp$ which is two dimensional Euclidean vector. The absolute value is to ensure that the square is positive.

Furthermore, we need to define the partonic differential cross section, which already incorporates the flux factor and an off-shell squared matrix element at tree-level.

$$d\Sigma_n^*(k_{\rm in}, k_{\overline{\rm in}}; \{p_i\}_{i=1}^n) = \left(\prod_{i=1}^n d^4p_i\delta(p_i^2 - m_i^2)\right)\delta^4\left(k_{\rm in} + k_{\overline{\rm in}} - \sum_{i=1}^n p_i\right)\frac{\left|\overline{\mathcal{M}}^*\right|^2(k_{\rm in}, k_{\overline{\rm in}}; \{p_i\}_{i=1}^n)}{4x\overline{x}P\cdot\overline{P}} \tag{33}$$

In that formula, the symbol n denotes the number of final-state particles, and the star in the superscript indicates that one of the initial states is off-shell. We assume that the symbol M always denotes a tree-level amplitude, and the squared matrix element present in equation (33) is averaged over initial-state spins and colors, while the final-state spins and colors are summed. In the list of momenta in both quantities, the semicolon is used to separate the initial-state partons from the final-state ones. Two initial states are

$$k_{\rm in}^\mu = xP^\mu + k_T^\mu, \quad k_{\overline{\rm in}}^\mu = \overline{x}\overline{P}^\mu. \tag{34}$$

Subsequently, we define a so called differential Born cross section. To do it we need to include a special function, named as jet function with the definition in equation (33). The role of the jet function is to ensure that the Born cross section is free from soft or collinear singularities. We define

$$d\mathrm{B}^\star\big(k_{\rm in}, k_{\overline{\rm in}}; \{p_i\}_n\big) = \frac{1}{\left|k_T\right|^2}d\Sigma_n^\star\big(k_{\rm in}, k_{\overline{\rm in}}; \{p_i\}_n\big)\,J_{\rm B}\big(\{p_i\}_n\big). \tag{35}$$

To ensure a smooth on-shell limit of the matrix element, we included the factor $1/|k_T|^2$ in the definition of the cross section. We also assume that the jet function $J_B\big(\{p_i\}_{i=1}^n\big)$ selects only parton momenta from the entire list of final states. The jet function determines the jets represented by a number of final-state partons. In our notation, it includes the decision wether there are enough jets. At Born level, the number of jets must be equal to the number of final-state partons, and this is indicated by the subscript B.

To define the real radiation part of the cross section at NLO, we need to introduce an advanced jet function $J_R\big(\{p_i\}_{i=1}^{n+1}\big)$, which accommodates one additional parton to be arbitrarily soft or allows one pair of partons to be collinear. Thus, this function produces a situation (referred to as the resolved situation) with one more jet than in the Born level cross section, as well as the unresolved situation where the jet function (the one with the subcript R) allows for one jet fewer than the number of final-state partons. We define the NLO *familiar real contribution* as

$$d\mathrm{R}^{\star\rm fam}\big(k_{\rm in}, k_{\overline{\rm in}}; \{p_i\}_{n+1}\big) = \frac{a_\epsilon\mu^{2\epsilon}}{\pi_\epsilon}\frac{1}{|k_T|^2}d\Sigma_n^\star\big(k_{\rm in}, k_{\overline{\rm in}}; \{p_i\}_{n+1}\big)\,J_{\rm R}\big(\{p_i\}_{i=1}^{n+1}\big) \tag{36}$$

where

$$\pi_\epsilon = \frac{\pi^{1-\epsilon}}{\Gamma(1-\epsilon)}, \quad a_\epsilon = \frac{\alpha_S}{2\pi}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}, \quad \epsilon = \frac{4-\mathrm{dim}}{2} \tag{37}$$

and μ is energy scale. The constant present in (36) is the correction factor (with comparison to Born level) which is needed due to application of the dimensional regularization and due to omitting the factors of 2π in the definition given by equation (33). We assume also coupling constant inside matrix element to be equal to unity.

We sometimes prefer to work with a positive parameter for dimensional regularization. Thus, we introduce

$$\bar{\epsilon} = -2\epsilon. \tag{38}$$

To summarize, to obtain the complete formula for the Born, real, and virtual contributions, one needs to include an overall factor of $(2\pi)^{4-3n} g_s^{2n} = (2\pi)^{4-3n}(4\pi\alpha_S)^n$.

With all the notation established in accordance with [1], we are ready to proceed to the NLO contributions in hybrid k_T-factorization. In this section we will begin with the real contribution, which involves additional phase space regions to be integrated. Subsequently in the following sections, we will address the virtual contribution, which entails one-loop amplitudes.

2.2 Familiar real contribution

In this subsection, we will study the divergent nature of the *familiar real contribution* defined in equation (36). We used the name *familiar* for the reasons that will become clear later. There are two divergences corresponding to the radiative gluon: one when it becomes soft, and the other when it becomes collinear to another parton.

We write the momenta of initial-state auxiliary parton as

$$k_1^\mu = -\Lambda P^\mu, \tag{39}$$

and final-state auxiliary parton as

$$k_2^\mu = p_\Lambda^\mu = (\Lambda - x)P^\mu - k_T^\mu + \frac{|k_T|^2}{(\Lambda - x)\cdot 2P\cdot\overline{P}}\overline{P}^\mu. \tag{40}$$

The soft limit was studied in [24, 25], and its universal structure in QCD is well-known. We will refer particularly to section 4.2 of [25], considering the terms relevant for quarks, as we chose them to be the auxiliary partons. Then we write only the terms involving the auxiliary partons

$$
\begin{aligned}
&- 2\sum_{i\neq q,\bar{q}}\left\{\frac{P\cdot p_i}{(P\cdot r)(r\cdot p_i)}(\overline{\mathcal{M}}^{\star}_{\text{aux-q}})^2_{iq} + \frac{P\cdot p_i}{(P\cdot r)(r\cdot p_i)}(\overline{\mathcal{M}}^{\star}_{\text{aux-q}})^2_{i\bar{q}}\right\} \\
&- 2\frac{\Lambda P\cdot p_\Lambda}{(\Lambda P\cdot r)(r\cdot p_\Lambda)}(\overline{\mathcal{M}}^{\star}_{\text{aux-q}})^2_{q\bar{q}},
\end{aligned}
\tag{41}
$$

where we omitted the overall constant and included the factor of 2 coming from the double counting over external momenta. The same result would be obtained by taking the soft limit on the amplitudes. Using the relation for color-correlated squared matrix elements (234), we obtain

$$
\frac{P\cdot p_i}{(P\cdot r)(r\cdot p_i)}(\overline{\mathcal{M}}^{\star}_{\text{aux-q}})^2_{iq} + \frac{P\cdot p_i}{(P\cdot r)(r\cdot p_i)}(\overline{\mathcal{M}}^{\star}_{\text{aux-q}})^2_{i\bar{q}} = \frac{xP\cdot p_i}{(xP\cdot r)(r\cdot p_i)}(\overline{\mathcal{M}}^{\star})^2_{i\star}, \tag{42}
$$

which has the form as if the space-like gluon were at the on-shell limit but with the space-like Born level. The last term in equation (41) straightforwardly vanishes in the limit $\Lambda\to\infty$ due to the fact that $\Lambda P\cdot p_\Lambda = -|k_T|^2/2 + \mathcal{O}(\Lambda^{-1})$. Nevertheless, we keep it for later explanation.

The ordinary collinear limit when the radiative gluon is collinear to the on-shell parton ($r\to\bar{x}_r\overline{P}$) is given by

$$
|\overline{\mathcal{M}}^{\star}|^2\Big(k_{\text{in}},\bar{x}\overline{P};r,\{p_i\}_n\Big)\to\frac{C_{\bar{x}}}{\overline{P}\cdot r}\frac{\mathcal{P}_{\bar{x}}(1-\bar{x}_r/\bar{x})}{\bar{x}-\bar{x}_r}|\overline{\mathcal{M}}^{\star}|^2\Big(k_{\text{in}},(\bar{x}-\bar{x}_r)\overline{P};\{p_i\}_n\Big), \tag{43}
$$

where $C_{\bar{x}}$ is the color factor taken out from the non-regularized splitting function $\mathcal{P}_{\bar{x}}$ associated with the on-shell initial state parton. Within the well-known factorization prescription, this singular behavior is absorbed into the NLO correction of the collinear PDF, as it fails to cancel the singularity arising from the virtual contribution. There is no need to discuss the final-state collinear singularities as they are the same as for the on-shell case.

Here we would like to addressed another singularity caused when the radiative gluon is collinear to the momentum P. We will derive this limit starting from the MHV amplitudes [26]. For the off-shell gluon $1^\star$ with a transverse momentum component, we have

$$
\mathcal{A}^{\star}_n(1^\star,2^+,3^-,4^+,\cdots,n^+) = \frac{1}{\kappa^*}\frac{\langle p_1p_3\rangle^4}{\langle p_1p_2\rangle\langle p_2p_3\rangle\langle p_3p_4\rangle\cdots\langle p_{n-1}p_n\rangle\langle p_np_1\rangle}, \tag{44}
$$

where we use the spinor notation introduced in appendix A and κ / κ^* defined in equations (315) and (318). Such an amplitude requires a factor of $|k_T|$ to avoid singularity in the limit $|k_T|\to 0$. We prefer to express the amplitude this way, and the necessary factor will be included while constructing the squared matrix element. We extract the limit by making the following substitutions: $p_1\to xP$ and $p_2\to yP$

$$
\begin{aligned}
&\mathcal{A}_n^\star(1^\star, 2^+, 3^-, 4^+, \cdots, n^+) \\
&\to \frac{1}{\langle p_1 p_2\rangle}\frac{1}{\kappa^*}\frac{x^2\langle Pp_3\rangle^4}{\sqrt{|y|}\langle Pp_3\rangle\langle p_3p_4\rangle\cdots\langle p_{n-1}p_n\rangle\sqrt{|x|}\langle p_nP\rangle} \\
&= \frac{1}{\langle p_1p_2\rangle}\frac{x^2}{\sqrt{|xy|}|x+y|}\frac{1}{\kappa^*}\frac{\langle (x+y)Pp_3\rangle^4}{\langle (x+y)Pp_3\rangle\langle p_3p_4\rangle\cdots\langle p_{n-1}p_n\rangle\sqrt{x}\langle p_n(x+y)P\rangle} \\
&= \frac{1}{\langle Pp_2\rangle}\frac{|x|}{\sqrt{|y|}|x+y|}\mathcal{A}_{n-1}^\star\big((x+y)P+k_T, 3^-, 4^+, \cdots, n^+\big).
\end{aligned}
\tag{45}
$$

For the opposite helicity, we obtain the same result with $\sqrt{|y|}/y$ instead of $1/y$ and $-1/[Pp_2]$. Denoting p_2 as the radiative gluon with momentum r, we obtain

$$
\begin{aligned}
\mathrm{Split}\Big(g^\star\big((x+y)P+k_T\big) \to g^\star\big(xP+k_T\big), g^+\big(yP\big)\Big) &= \frac{1}{\langle Pr\rangle}\frac{|x|}{\sqrt{|y|}|x+y|} \\
\mathrm{Split}\Big(g^\star\big((x+y)P+k_T\big) \to g^\star\big(xP+k_T\big), g^-\big(yP\big)\Big) &= \frac{-1}{[Pr]}\frac{\mathrm{sgn}(y)|x|}{\sqrt{|y|}|x+y|}.
\end{aligned}
\tag{46}
$$

For off-shell squared matrix elements this cause the collinear singularity. In the limit $r \to x_r P$ we find

$$
|\overline{\mathcal{M}}^\star|^2\Big(x, k_T, \overline{x}; r, \{p_i\}_n\Big) \to 2N_c\frac{1}{P\cdot r}\frac{x^2}{x_r(x-x_r)^2}|\overline{\mathcal{M}}^\star|^2\Big((x-x_r), k_T, \overline{x}; \{p_i\}_n\Big). \tag{47}
$$

We will adapt this formula to the situation where the radiative gluon is not exactly collinear to $\overline{P}$. However, working within the k_T-factorization, we are allowed to use the transverse momentum in the initial state for the recoil. Thus, we rewrite equation (47) as

$$
\begin{aligned}
&|\overline{\mathcal{M}}^\star|^2\Big(xP+k_T, k_{\overline{\mathrm{in}}}; r, \{p_i\}_n\Big) \\
&\qquad \to \frac{2N_c}{P\cdot r}\frac{x^2}{x_r(x-x_r)^2}|\overline{\mathcal{M}}^\star|^2\Big((x-x_r)P+k_T-r_T, (\overline{x}-\overline{x}_r)\overline{P}; \{p_i\}_n\Big).
\end{aligned}
\tag{48}
$$

Where $\{p_i\}_n$ on the right-hand side is the same as on the left-hand side for any radiation r. We will make a slight modification by neglecting $\overline{x}_r$ on the right-hand side, which will not affect the correctness of the limit. This is justified by the light-cone condition $\overline{x}_r = \frac{|r_T|^2}{2P\cdot\overline{P}x_r}$, under which $\overline{x}_r$ approaches 0 as $|r_T|^2$ in the limit $r_T \to 0$. Therefore, as r_T diminishes, $\overline{x}_r$ becomes negligible, allowing us to simplify the expression without losing accuracy in this limit. In the following we would like to briefly derive the formula for the cross section with this divergence. The more detailed discussion we will address in section 3.

Before we proceed, we would like to state that we restrict the phase space to region B of figure 6, limited by equations (27) with b $= x$. Hence, we will take into account the relevant collinear and soft regions. Thus, the phase space restriction is given by

$$x_r > \frac{|r_T|}{|k_T - r_T|}, \tag{49}$$

in the collinear limit, i.e. $r_T \to 0$, this restriction is the complement to the one from equation (267). Thanks to this, we include the entire phase space in the relevant real contribution and avoid double counting.

We insert equation (48) into the cross section formula (36). We also write the normalization of x_r and $\overline{x}_r$ explicitly and we chose $D(r_T)$ to be

$$D(r_T) = \frac{|k_T - r_T|^2}{\nu^2 x\overline{x}}. \tag{50}$$

With the correction for the flux factor, we find

$$\begin{aligned}
&dR^{\star\text{fam}}_{\text{coll}}\big(k_{\text{in}}, k_{\overline{\text{in}}}; \{p_i\}_n\big)\\
&= \frac{a_\epsilon}{\pi_\epsilon \mu^{\overline{\epsilon}}} \frac{1}{|k_T|^2} \int d^{4+\overline{\epsilon}} r \delta(r^2) \theta\big(x_r < x\big) \theta\Big(\frac{\overline{x}_r}{\overline{x}} < D(r_T)\frac{x_r}{x}\Big) 2N_c \frac{1}{P\cdot r} \frac{x^2}{x_r(x-x_r)^2}\\
&\qquad\qquad \times \frac{x - x_r}{x} d\Sigma^\star_n\big(x - x_r, k_T - r_T, \overline{x}; \{p_i\}_n\big) J_{\text{B}}\big(\{p_i\}_n\big)\\
&\qquad = \frac{2a_\epsilon N_c}{\pi_\epsilon \mu^{\overline{\epsilon}}} \int_0^x \frac{dx_r}{x_r} \int \frac{d^{2+\overline{\epsilon}} r_T}{|r_T|^2} \frac{|k_T - r_T|^2}{|k_T|^2} \theta\Big(|r_T| < |k_T - r_T|\frac{x_r}{x}\Big)\\
&\qquad\qquad \times \frac{x}{x - x_r} d\text{B}^\star\big(x - x_r, k_T - r_T, \overline{x}; \{p_i\}_n\big),
\end{aligned} \tag{51}$$

where we used the relation $P \cdot r = |r_T|^2/(2x_r)$. We multiply this by special function $F(x, k_T)$ and integrate over x and k_T. We obtain

$$\begin{aligned}
&\int_0^1 \frac{dx}{x} \int d^2k_T F(x, k_T) dR^{\star\text{fam}}_{\text{coll}}\big(k_{\text{in}}, k_{\overline{\text{in}}}; \{p_i\}_n\big)\\
&= \frac{2a_\epsilon N_c}{\pi_\epsilon \mu^{\overline{\epsilon}}} \int_0^1 \frac{dy}{y} \int d^2k_T \int_y^1 \frac{dz}{z(1-z)} \int \frac{d^{2+\overline{\epsilon}} r_T}{|r_T|^2} \frac{|k_T|^2}{|k_T + r_T|^2} \theta\big(|r_T| < |k_T|(1-z)\big)\\
&\qquad\qquad \times F\Big(\frac{y}{z}, k_T + r_T\Big) d\text{B}^\star\big(y, k_T\, \overline{x}; \{p_i\}_n\big)\\
&= \int_0^1 \frac{dx}{x} \int d^2k_T \tilde{F}(x, k_T) d\text{B}^\star\big(x, k_T\, \overline{x}; \{p_i\}_n\big),
\end{aligned} \tag{52}$$

with

$$\tilde{F}(x, k_T) = \frac{2a_\epsilon N_c}{\pi_\epsilon \mu^{\overline{\epsilon}}} \int_x^1 \frac{dz}{z(1-z)} \int_0^{|k_T|(1-z)} \frac{d^{2+\overline{\epsilon}} r_T}{|r_T|^2} \frac{|k_T|^2}{|k_T + r_T|^2} F\Big(\frac{x}{z}, k_T + r_T\Big), \tag{53}$$

where we applied the notation given by

$$\int_0^U d^{2+\bar{\epsilon}} r_T = \int d^{2+\bar{\epsilon}} r_T \theta\big(|r_T| < U\big). \tag{54}$$

To arrive at equation (52) we performed the variable substitution $x_r \to zx$ and $x \to y/(1-z)$.

We find equation (53) as formula (2.21) in [27], which is recognized as the real-emission term of the multi-Regge kinematics evolution equation. With the redefinition of the function $F(x, k_T) \to |k_T|^2 F(x, k_T)$, we can eliminate the factor $\frac{|k_T|^2}{|k_T + r_T|^2}$, which is not present in [27]. Nevertheless, we prefer to adhere to the equation given above. Realizing that

$$F(x, 0) = 0, \tag{55}$$

we see that there is no singularity when $r_T = -k_T$. We add and subtract a term in equation (53), which allow us to isolate the $1/\bar{\epsilon}$ divergences, we write

$$\begin{aligned}\tilde{F}(x, k_T) = \frac{2a_\epsilon N_c}{\pi_\epsilon \mu^{\bar{\epsilon}}} \int_x^1 \frac{dz}{z(1-z)} \int_0^{|k_T|(1-z)} \frac{d^{2+\bar{\epsilon}} r_T}{|r_T|^2} \frac{|k_T|^2}{|k_T + r_T|^2} \\ \times \left[F\Big(\frac{x}{z}, k_T + r_T\Big) - \frac{|r_T + k_T|^2}{|k_T|^2} F\Big(\frac{x}{z}, k_T\Big) \right] \\ + \tilde{F}^{\text{div}}(x, k_T),\end{aligned} \tag{56}$$

with all poles restored in

$$\begin{aligned}\tilde{F}^{\text{div}}(x, k_T) &= \frac{2a_\epsilon N_c}{\pi_\epsilon \mu^{\bar{\epsilon}}} \int_x^1 \frac{dz}{z(1-z)} \int_0^{|k_T|(1-z)} \frac{d^{2+\bar{\epsilon}} r_T}{|r_T|^2} F\Big(\frac{x}{z}, k_T\Big) \\ &= \frac{4a_\epsilon N_c}{\bar{\epsilon}} \left(\frac{|k_T|}{\mu} \right)^{\bar{\epsilon}} \int_x^1 \frac{dz}{z(1-z)^{1-\bar{\epsilon}}} F\Big(\frac{x}{z}, k_T\Big) \\ &= \frac{4a_\epsilon N_c}{\bar{\epsilon}} \left(\frac{|k_T|}{\mu} \right)^{\bar{\epsilon}} \int_x^1 dz F\Big(\frac{x}{z}, k_T\Big) \left\{ \frac{(1-z)^{\bar{\epsilon}}}{z} + \frac{(1-z)^{\bar{\epsilon}}}{(1-z)} \right\} \\ &= \frac{4a_\epsilon N_c}{\bar{\epsilon}} \left(\frac{|k_T|}{\mu} \right)^{\bar{\epsilon}} \left\{ \int_x^1 dz F\Big(\frac{x}{z}, k_T\Big) \left[\frac{(1-z)^{\bar{\epsilon}}}{z} + \frac{(1-z)^{\bar{\epsilon}}}{[1-z]_+} \right] \right. \\ &\qquad \left. + \int_0^1 dz F\Big(x, k_T\Big) \frac{(1-z)^{\bar{\epsilon}}}{(1-z)} \right\},\end{aligned} \tag{57}$$

and finally

$$\begin{aligned}\tilde{F}^{\text{div}}(x, k_T) = 4a_\epsilon N_c \left(\frac{|k_T|}{\mu} \right)^{\bar{\epsilon}} \left\{ \int_x^1 dz F\Big(\frac{x}{z}, k_T\Big) \left[\frac{\ln(1-z)}{z} + \frac{\ln(1-z)}{[1-z]_+} \right] \right. \\ + \frac{1}{\bar{\epsilon}} \int_x^1 dz F\Big(\frac{x}{z}, k_T\Big) \left[\frac{1}{z} + \frac{1}{[1-z]_+} \right] \\ \left. + \frac{1}{\bar{\epsilon}^2} F\Big(x, k_T\Big) + \mathcal{O}(\bar{\epsilon}) \right\},\end{aligned} \tag{58}$$

where

$$\int_x^1 \frac{dz}{[1-z]_+} f(z) = \int_0^1 \frac{dz}{(1-z)} \Big[f(z)\theta(x<z) - f(1) \Big]. \tag{59}$$

The $1/\bar{\epsilon}^2$ correspond to soft divergency, and this term cancel against the similar term from virtual contribution. Please note that equation (58) is the correct formula for equation (6.14) in [1], where the braces are incorrect. The second line of equation (58) is very similar to the corresponding initial-state collinear divergence known from collinear factorization. This is the only difference in divergences for real contributions for collinear and k_T-factorization.

3 Divergences in real contribution

In the previous section, we established the real contribution and carefully addressed all divergent terms. In order to have correct NLO contribution, we need to have those divergences under control. Fortunately, they are process-independent providing the perspective of their treatment within a renomalization procedure. We also observed that the process-dependent parts align with those found in collinear factorization, with one exception, which is a divergence specifically related to the space-like initial state.

The subtraction method offers a powerful approach to regularize divergent integrals, enabling the use of numerical methods for their evaluation. In this section, we apply the subtraction method to handle the real-radiation integrals, allowing us to derive universal formulas for the divergent parts. Given that we are working with an off-shell initial state gluon that possesses non-zero transverse components, we can effectively subtract the momentum recoil from it. This approach ensures that the final state momenta remain largely unaffected.

We organized the subtraction terms with the approach provided in [5, 6]. The whole work and results are addressed in [2]. Here, we provide only the main results corresponding to the divergences discussed in section 2.

3.1 Notation

In this subsection we will briefly introduce some necessary new notation. First of all we would like to stress out that we work in the frame for which

$$P^\mu = (E, 0, 0 - E), \quad \overline{P}^\mu = (\overline{E}, 0, 0 - \overline{E}), \quad \text{and} \quad S = 2P \cdot \overline{P} = 4E\overline{E}, \tag{60}$$

where E is the energy of the momentum and this symbol will be used for any momentum with the specification by subscript. We will also apply the unit vector $n^2 = 0$. Hence, for light-like momentum we can write

$$K^\mu = E_K n_K^\mu = E_K(1, \vec{n}_K). \tag{61}$$

We will also introduce one symbol for the sum of the initial-state momenta

$$Q^\mu = k_{\text{in}}^\mu + k_{\overline{\text{in}}}^\mu = xP^\mu + \overline{x}\overline{P}^\mu + k_T^\mu, \tag{62}$$

and

$$\int [dQ] = \int_0^1 dx \int_0^1 d\overline{x} \int d^2k_T. \tag{63}$$

We writhe the PDF's as one function with the flux factor

$$\mathcal{L}\Big(Q;\{p_i\}_n\Big) = \frac{F\big(x,k_T,\mu_F(\{p_i\}_n)\big)\,f\big(\overline{x},\mu_F(\{p_i\}_n)\big)}{2x\overline{x}S}, \tag{64}$$

where μ_F is the factorization scale. We also define

$$d\Phi\Big(Q;\{p_i\}_n\Big) = \left(\prod_{i=1}^{n}\frac{d^4p_i}{(2\pi)^3}\delta_+(p_i^2-m_i^2)\right)\frac{1}{(2\pi)^4}\delta\Big(Q-\sum_{i=0}^{n}p_i\Big). \tag{65}$$

With those we can rewrite the definition from equation (28). Thus, we write the formula for the Born-level k_T-factorization cross section

$$\sigma_B = \frac{1}{\mathcal{S}_n}\int[dQ]\int d\Phi\Big(Q;\{p_i\}_n\Big)\mathcal{L}\Big(Q;\{p_i\}_n\Big)|\mathcal{M}|^2\Big(Q;\{p_i\}_n\Big)J_B\Big(\{p_i\}_n\Big), \tag{66}$$

where $\mathcal{S}_n$ consist of symmetry factors or of average over initial-state degrees of freedom factors.

Later we will use the following notation concerning the color factors.

$$C_i = \begin{cases} C_g = C_A = N_c & \text{if } i \text{ refer to a gluon,} \\ C_q = C_F = \dfrac{N_c-1}{2N_c} & \text{if } i \text{ refer to a quark or antiquark,} \end{cases} \tag{67}$$

and

$$C_{ij} = \begin{cases} C_g & \text{if both } i,\, j \text{ refer to a gluons,} \\ C_q & \text{if exactly one of } i,\, j \text{ refers to a gluon,} \\ T_R & \text{if none of } i,\, j \text{ refers to a gluon,} \end{cases} \tag{68}$$

and $T_R = \frac{1}{2}$. Color conservation implies

$$\sum_i(\mathcal{M})^2_{\mathrm{color}(i,b)} = 0 \Leftrightarrow \sum_{b\neq i}(\mathcal{M})^2_{\mathrm{color}(i,b)} = -C_i|\mathcal{M}^\star|^2, \tag{69}$$

where $(\mathcal{M})_{\mathrm{color}(i,b)}$ is the color correlated matrix element described in detail in appendix D.

3.2 Types of the divergences

In the real radiation contribution at NLO, the associated jet function J_R allows partons to cause singularities. We assume that all final-state partons are light-like. Specifically, we can encounter situations where a pair of partons becomes collinear

$$p_r \parallel p_i \quad \Leftrightarrow \quad \vec{n}_r - \vec{n}_i \to \vec{0}, \tag{70}$$

or where a single parton becomes soft

$$p_r \to \text{soft} \quad \Leftrightarrow \quad E_r \to 0. \tag{71}$$

Then the J_R behaves as

$$
\begin{aligned}
J_R\Big(\{p_i\}_{n+1}\Big) &\xrightarrow{p_r\to\text{soft}} J_B\Big(\{p_i\}_n^{\not r}\Big),\\
J_R\Big(\{p_i\}_{n+1}\Big) &\xrightarrow{p_r\|p_i} J_B\Big(\{p_i\}_n^{\not r;i}\Big),\\
J_R\Big(\{p_i\}_{n+1}\Big) &\xrightarrow{p_r\|P,\overline{P}} J_B\Big(\{p_i\}_n^{\not r}\Big),
\end{aligned}
\tag{72}
$$

where removing momentum p_r from $\{p_i\}_{n+1}$ results in $\{p_i\}_n^{\not r}$, while removing momentum p_r and replacing $p_i \to (1+z_{ri})p_i$ in $\{p_i\}_{n+1}$ results in $\{p_i\}_n^{\not r;i}$, with

$$
z_{ri} = \frac{E_r}{E_i},
\tag{73}
$$

and if $p_r \parallel p_i$, we also have

$$
(1+z_{ri})p_i = p_r + p_i.
\tag{74}
$$

We are now ready to introduce the limits of the off-shell matrix element that cause the singularities. Fortunately, it can be expressed in a factorized form. Within hybrid k_T-factorization, we can redistribute the recoil over the initial-state momenta. We write the limits as

$$
\begin{aligned}
\text{soft}\quad & |\mathcal{M}|^2\Big(Q;\{p_i\}_{n+1}\Big) \xrightarrow{p_r\to\text{soft}} \mathcal{R}^{\text{soft}}(p_r)\mathcal{A}^{\text{soft}}(p_r)(Q-p_r;\{p_i\}_n^{\not r}),\\
\text{final coll}\quad & |\mathcal{M}|^2\Big(Q;\{p_i\}_{n+1}\Big) \xrightarrow{p_r\|p_i} \mathcal{R}_{ir}^{\text{F,coll}}(p_r)\mathcal{A}_{ir}^{\text{F,coll}}(p_r)(Q-p_r+z_{ri}p_i;\{p_i\}_n^{\not r;i}),\\
\text{initial coll}\quad & |\mathcal{M}|^2\Big(Q;\{p_i\}_{n+1}\Big) \xrightarrow{p_r\|P} \mathcal{R}_{xr}^{\text{I,coll}}(p_r)\mathcal{A}_{xr}^{\text{I,coll}}(p_r)(Q-p_r;\{p_i\}_n^{\not r}),\\
\text{initial }\overline{\text{coll}}\quad & |\mathcal{M}|^2\Big(Q;\{p_i\}_{n+1}\Big) \xrightarrow{p_r\|\overline{P}} \mathcal{R}_{\bar{x}r}^{\text{I,coll}}(p_r)\mathcal{A}_{\bar{x}r}^{\text{I,coll}}(p_r)(Q-p_r;\{p_i\}_n^{\not r}).
\end{aligned}
\tag{75}
$$

The singular behavior of the matrix element is represented by $\mathcal{R}$ as a function of p_r, while the quantity $\mathcal{A}$ consists of the squared tree-level amplitudes, including color and spin correlators. The quantities are well-defined across the entire phase space. Their role is to construct the subtraction terms. However, we must account for the double counting of the soft-collinear singularities.

We define the finite real radiation integral as

$$
\begin{aligned}
\sigma_R^{\text{fin}} = \frac{1}{\mathcal{S}_{n+1}}\int[dQ]\int d\Phi\Big(Q;\{p_i\}_{n+1}\Big)\Bigg\{&\mathcal{L}\Big(Q;\{p_i\}_{n+1}\Big)|\mathcal{M}|^2\Big(Q;\{p_i\}_{n+1}\Big)J_R\Big(\{p_i\}_{n+1}\Big)\\
&-\sum_r \text{Subt}_r\Big(Q;\{p_i\}_{n+1}\Big)\Bigg\}
\end{aligned}
\tag{76}
$$

which has one more final state compared to the Born level cross section and the sum runs over all final-state momenta. For that integral, the jet function allows a pair of momenta to become collinear or for one parton to become arbitrarily soft. This results in the divergence of the integral. To handle these singularities, we construct the subtraction term in such a way that the integration of the subtracted real radiation integral can be performed numerically. For each final-state parton with momentum p_r there is a subtraction term given by

$$
\begin{aligned}
&\mathrm{Subt}_r\Big(Q;\{p_i\}_{n+1}\Big) = \\
&\quad \sum_i \mathcal{L}\Big(Q-q_{r,i};\{p_i\}_n^{\not r;i}\Big)\mathcal{R}^{\mathrm{F}}_{ir}(p_r)\mathcal{A}^{\mathrm{F}}_{ir}\Big(Q-p_r+z_{ri}p_i;\{p_i\}_n^{\not r;i}\Big)J_B\Big(\{p_i\}_n^{\not r;i}\Big) \\
&\quad + \sum_{a\in\{k_{\mathrm{in}},k_{\overline{\mathrm{in}}}\}} \mathcal{L}\Big(Q-q_{r,a};\{p_i\}_n^{\not r}\Big)\mathcal{R}^{\mathrm{I,coll}}_{ar}(p_r)\mathcal{A}^{\mathrm{I,coll}}_{ar}\Big(Q-p_r;\{p_i\}_n^{\not r}\Big)J_B\Big(\{p_i\}_n^{\not r}\Big) \\
&\quad + \sum_{a\in\{k_{\mathrm{in}},k_{\overline{\mathrm{in}}}\}} \mathcal{L}\Big(Q-q_r;\{p_i\}_n^{\not r}\Big)\mathcal{R}^{\mathrm{I,soft}}_{a}(p_r)\mathcal{A}^{\mathrm{I,soft}}_{a}\Big(Q-p_r;\{p_i\}_n^{\not r}\Big)J_B\Big(\{p_i\}_n^{\not r}\Big) \\
&\quad + \sum_{a\in\{k_{\mathrm{in}},k_{\overline{\mathrm{in}}}\}} \mathcal{L}\Big(Q-q_r;\{p_i\}_n^{\not r}\Big)\mathcal{R}^{\mathrm{I,soco}}_{a}(p_r)\mathcal{A}^{\mathrm{I,soco}}_{a}\Big(Q-p_r;\{p_i\}_n^{\not r}\Big)J_B\Big(\{p_i\}_n^{\not r}\Big),
\end{aligned}
\tag{77}
$$

where the i-sum runs over all final-state momenta and $\mathcal{R}^{\mathrm{F}}_{rr}(p_r)=0$. The terms with F and I refer to final and initial singularities, respectively. The label r denotes the dependence on the flavor of the final-state parton. The first three lines illustrate the three types of momentum recoils present in our subtraction scheme.

The subtracted momentum q is different in each subtraction term and vanishes in the limit for which the given subtraction term cures the singularity. For an initial soft singularity, $q=p_r$, but for an initial collinear singularity, this is not possible as p_r does not vanish and the value of $\mathcal{L}\Big(Q-p_r;\{p_i\}_n^{\not r}\Big)$ would not match $\mathcal{L}\Big(Q;\{p_i\}_n^{\not r}\Big)$ at the singular limit. In other cases, q can be identical to the recoil from the matrix element.

The integration of the $\sum_r \mathrm{Subt}_r\Big(Q;\{p_i\}_{n+1}\Big)$ should produce the divergent terms that match the divergences of the virtual contribution addressed in section 4. We show in the appendix D of [2], that the subtraction term, with the variable substitution given by equation (94), integrate to

$$
\begin{aligned}
\sigma_R^{\mathrm{div}} =& \frac{1}{\mathcal{S}_{n+1}}\sum_r\int[dQ]\int d\Phi\Big(Q;\{p_i\}_n^{\not r}\Big)J_B\Big(\{p_i\}_n^{\not r}\Big) \\
&\times\Bigg\{\sum_i \mathcal{L}^{\mathrm{F}}_{ir}\Big(\epsilon,Q,\{p_i\}_n^{\not r}\Big)\mathcal{A}^{\mathrm{F}}_{ir}\Big(Q;\{p_i\}_n^{\not r}\Big)+\sum_{a\in\{k_{\mathrm{in}},k_{\overline{\mathrm{in}}}\}}\mathcal{L}^{\mathrm{I}}_{ar}\Big(\epsilon,Q,\{p_i\}_n^{\not r}\Big)\mathcal{A}^{\mathrm{I}}_{ar}\Big(Q;\{p_i\}_n^{\not r}\Big)\Bigg\},
\end{aligned}
\tag{78}
$$

where

$$
\mathcal{L}^{\mathrm{I}}_{ar}\mathcal{A}^{\mathrm{I}}_{ar} = \mathcal{L}^{\mathrm{I,coll}}_{ar}\mathcal{A}^{\mathrm{I,coll}}_{ar}+\mathcal{L}^{\mathrm{I,soft}}_{ar}\mathcal{A}^{\mathrm{I,soft}}_{ar}+\mathcal{L}^{\mathrm{I,soco}}_{ar}\mathcal{A}^{\mathrm{I,soco}}_{ar}, \tag{79}
$$

and

$$\mathcal{L}^{\mathrm{F}}_{ir}\Big(\epsilon, Q, \{p_i\}^{\not{f}}_n\Big) = \int \frac{d^{4-2\epsilon}p_r}{(2\pi)^{3-2\epsilon}}\delta_+(p_r^2)(1-z_{ri})\mathcal{R}^{\mathrm{F}}_{ir}(p_r)\Theta(p_r - z_{ri}p_i) \times \mathcal{L}\Big(Q + p_r - z_{ri}p_i - q_{r,i}; \{p_i\}^{\not{f}}_n\Big), \tag{80}$$

$$\mathcal{L}^{\mathrm{I,coll}}_{ar}\Big(\epsilon, Q, \{p_i\}^{\not{f}}_n\Big) = \int \frac{d^{4-2\epsilon}p_r}{(2\pi)^{3-2\epsilon}}\delta_+(p_r^2)\mathcal{R}^{\mathrm{I,coll}}_{ar}(p_r)\Theta(p_r)\mathcal{L}\Big(Q + p_r - q_{r,a}; \{p_i\}^{\not{f}}_n\Big), \tag{81}$$

$$\mathcal{L}^{\mathrm{I,soft/soco}}_{ar}\Big(\epsilon, Q, \{p_i\}^{\not{f}}_n\Big) = \int \frac{d^{4-2\epsilon}p_r}{(2\pi)^{3-2\epsilon}}\delta_+(p_r^2)\mathcal{R}^{\mathrm{I,soft/soco}}_{a}(p_r)\Theta(p_r)\mathcal{L}\Big(Q + p_r - q_r; \{p_i\}^{\not{f}}_n\Big), \tag{82}$$

with

$$\Theta(q) = \theta(-x < x_q < 1 - x)\theta(-\overline{x} < \overline{x}_q < 1 - \overline{x}) \tag{83}$$

Thus, we can write

$$\sigma_R = \sigma_R^{\mathrm{fin}} + \sigma_R^{\mathrm{div}}. \tag{84}$$

From these equations, we can realize that the best choice for q is the recoil momenta, which results in $\mathcal{L}$ being independent of the integration variable. Nevertheless, for the initial-state, it is not possible to avoid the arguments $\mathcal{L}\big(Q + x_r P; \{p_i\}^{\not{f}}_n\big)$ and $\mathcal{L}\big(Q + \overline{x}_r \overline{P}; \{p_i\}^{\not{f}}_n\big)$.

Let us introduce the quantities $L_Z^{X,Y}$, which will be used for handling the integrated subtraction term from equation (78). We write

$$\mathcal{L}^{\mathrm{F}}_{ir}\mathcal{A}^{\mathrm{F}}_{ir} = a_\epsilon \mathcal{L}\Big(Q; \{p_i\}_n\Big)\Big[L^{\mathrm{F,coll}}_{ir}|\mathcal{M}_{ir}|^2 + \sum_b L^{\mathrm{F,soft}}_{ib}(\mathcal{M})^2_{\mathrm{color}(i,b)} + L^{\mathrm{F,soco}}_{i}|\mathcal{M}|^2\Big], \tag{85}$$

$$\mathcal{L}^{\mathrm{I}}_{ar}\mathcal{A}^{\mathrm{I}}_{ar} = a_\epsilon \mathcal{L}\Big(Q; \{p_i\}_n\Big)\Big[L^{\mathrm{I,coll}}_{ar}|\mathcal{M}_{ar}|^2 + \sum_b L^{\mathrm{I,soft}}_{ab}(\mathcal{M})^2_{\mathrm{color}(a,b)} + L^{\mathrm{I,soco}}_{a}|\mathcal{M}|^2\Big]. \tag{86}$$

Then we can decompose each $L_Z^{X,Y}$ into a finite part, which must be evaluated using numerical integration, and a divergent part, which can be calculated analytically. Hence, the decomposition is

$$L_Z^{X,Y}(\epsilon) = L_Z^{X,Y,\mathrm{div}}(\epsilon) + L_Z^{X,Y,\mathrm{fin}} + \mathcal{O}(\epsilon). \tag{87}$$

To perform the numerical integration, the integration space for the Monte Carlo integral implied by $d\Phi\Big(Q; \{p_i\}^{\not{f}}_n\Big)$ from equation (78) can simply be augmented with the necessary extra variables. Then, this integration is not computationally much expensive.

We are now ready to define and calculate the final-state and initial-state parts of the subtraction term. This issue will be addressed in the following subsections. When evaluating the subtraction scheme, one will encounter the arbitrary parameters E_0, ζ_0, ξ_0.

3.3 Final-state terms

In equation (77), we refer to the terms with arguments $\left(Q - p_r + z_{ri}p_i; \{p_i\}_n^{\not r;i}\right)$ as final-state terms, even though they involve initial-state spectators. The final-state terms are given by

$$\mathcal{R}_{ir}^{\mathrm{F}}\mathcal{A}_{ir}^{\mathrm{F}} = \mathcal{R}_{ir}^{\mathrm{F,coll}}\mathcal{A}_{ir}^{\mathrm{F,coll}} + \mathcal{R}_{i}^{\mathrm{F,soft}}\mathcal{A}_{i}^{\mathrm{F,soft}} + \mathcal{R}_{i}^{\mathrm{F,soco}}\mathcal{A}_{i}^{\mathrm{F,soco}}, \tag{88}$$

where

$$\mathcal{R}_{ir}^{\mathrm{F,coll}}\mathcal{A}_{ir}^{\mathrm{F,coll}} = \frac{4\pi\alpha_s}{\mu^{\bar{\epsilon}}}\theta(n_r \cdot n_i < 2\zeta_0)\frac{\theta(E_r < E_i)}{p_i \cdot p_r}\mathcal{Q}_{ir}(z_{ri})|\mathcal{M}_{ir}|^2, \tag{89}$$

$$\mathcal{R}_{i}^{\mathrm{F,soft}}\mathcal{A}_{i}^{\mathrm{F,soft}} = -\frac{4\pi\alpha_s}{\mu^{\bar{\epsilon}}}\theta(E_r < E_0)\frac{2}{n_i \cdot p_r}\sum_b \frac{n_i \cdot n_b}{n_i \cdot p_r + n_b \cdot p_r}(\mathcal{M})^2_{\mathrm{color}(i,b)}, \tag{90}$$

$$\mathcal{R}_{i}^{\mathrm{F,soco}}\mathcal{A}_{i}^{\mathrm{F,soco}} = -\frac{4\pi\alpha_s}{\mu^{\bar{\epsilon}}}\theta(E_r < E_0)\theta(n_r \cdot n_i < 2\zeta_0)\frac{2C_i}{p_i \cdot p_r}\frac{1}{z_{ri}}|\mathcal{M}|^2. \tag{91}$$

The function $\mathcal{Q}_{ir}(z_{ri})$ from equation (89) is a collinear splitting function, but formulated in terms of the ratio of the energies of the splitting products instead of the usual splitting variable, which is the fraction of the original energy carried by the radiator after splitting.

The b-sum runs over both initial-state and final-state partons, with $n_i \cdot n_i = 0$. The argument mentioned above applies to all of these matrix elements. The term labeled "soco" refers to the soft-collinear term, which corrects the double counting. If r does not refer to a gluon, then both the soft and soft-collinear terms vanish. In the soft limit $E_r \to 0$, the soft-collinear and collinear terms cancel, while in the collinear limit, the soft and soft-collinear terms cancel. This avoids double counting, and the phase space restrictions $\theta(E_r < E_0)$ do not interfere with this cancellation. The result should not depend on the parameters E_0 and ζ_0. While these parameters are somewhat arbitrary, they provide a valuable means for cross-checking correctness and allow for adjustments to the numerical integration. Nevertheless, σ_R^{fin} and the finite part of σ_R^{div} individually depend on the parameters, but their sum does not.

We find it beneficial to split the soft factor into two terms, each with only one collinear singularity.

$$\frac{(n_a \cdot n_b)}{(n_a \cdot p_r)(p_r \cdot n_b)} = \frac{1}{n_a \cdot p_r}\frac{n_a \cdot n_b}{n_a \cdot p_r + p_r \cdot n_b} + \frac{1}{p_r \cdot n_b}\frac{n_a \cdot n_b}{n_a \cdot p_r + p_r \cdot n_b}. \tag{92}$$

We mention at this point that for final-state collinear terms, we are able to avoid double counting, thanks to the function $\theta(E_r < E_i)$, which ensures that the radiation has lower energy than the radiator.

Our goal is to combine the equations (80) and (88) and also the functions given by equations

(83). We will start with the collinear terms.

3.3.1 Collinear terms

Hence, with equation (89) we have

$$\begin{aligned} L_{ir}^{\mathrm{F,coll}} &= \frac{1}{\pi_\epsilon \mu^{\bar{\epsilon}}} \int d^{4+\bar{\epsilon}} p_r \delta_+(p_r^2)(1-z_{ri})\Theta(p_r - z_{ri}p_i) \\ &\qquad \times \theta(n_r \cdot n_i < 2\zeta_0) \frac{\theta\big(z_{ri} < \frac{1}{2}\big)}{(1-z_{ri})p_i \cdot p_r} \mathcal{P}_{ir}(1-z_{ri}) \\ &= \frac{1}{\pi_\epsilon \mu^{\bar{\epsilon}}} \int d^{4+\bar{\epsilon}} p_r \delta_+(p_r^2) \mathcal{S}^{\mathrm{F,coll}}(p_r;p_i)\Theta(p_r - z_{ri}p_i). \end{aligned} \tag{93}$$

where the variables z_{ri} and p_i differ from those of the equation (89) by

$$\begin{aligned} z_{ri} &\to \frac{z_{ri}}{1-z_{ri}}, \\ p_i &\to (1-z_{ri})p_i. \end{aligned} \tag{94}$$

The function $\mathcal{S}^{\mathrm{F,coll}}(p_r;p_i)$ defined by

$$\mathcal{S}^{\mathrm{F,coll}}(p_r;p_i) = \frac{\theta(n_r \cdot n_i < 2\zeta_0)\theta\big(z_{ri} < \frac{1}{2}\big)}{p_i \cdot p_r} \mathcal{P}_{ir}(1-z_{ri}), \tag{95}$$

which is singular in the collinear limit $\vec{n}_r \to \vec{n}_i$, and due to the $1/z_{ri}$ factor from $\mathcal{P}_{ir}(1-z_{ri})$, the function $\mathcal{S}^{\mathrm{F,coll}}(p_r;p_i)$ is also singular in the soft limit. Nevertheless, $p_r - z_{ri}p_i$ vanishes in both limits. Here, $\mathcal{P}_{ir}(1-z_{ri})$ is the typical splitting function, as opposed to $\mathcal{Q}_{ir}$ we introduced earlier in equation (89). We realize that the integral

$$L_{ir}^{\mathrm{F,coll,fin}} = \frac{1}{\pi} \int d^4 p_r \delta_+(p_r^2) \mathcal{S}^{\mathrm{F,coll}}(p_r;p_i) \Big[\Theta(p_r - z_{ri}p_i) - 1\Big], \tag{96}$$

is finite and is to be calculated numerically together with equation (78) by constructing one momentum p_r for every $\{p_i\}_n^{\not{f}}$ in equation (78) and evaluating the integrand of equation (96) once for that p_r.

Then the divergent part

$$L_{ir}^{\mathrm{F,coll,div}}(\epsilon) = \frac{1}{\pi_\epsilon \mu^{\bar{\epsilon}}} \int d^{4+\bar{\epsilon}} p_r \delta_+(p_r^2) \mathcal{S}^{\mathrm{F,coll}}(p_r;p_i). \tag{97}$$

is to be calculated analytically. The Θ function from equation (93) makes it unnecessarily complicated for the integral to be performed completely analytically.

3.3.2 Soft terms

As mentioned before, $p_{rT} - z_{ri}p_{iT}$ vanishes in soft limit, then the integral

$$
\begin{aligned}
L_{ib,\mathrm{compl}}^{\mathrm{F,soft,fin}} = \frac{-2}{\pi}\int d^4p_r\delta_+(p_r^2)\frac{1}{n_i\cdot p_r}\frac{n_i\cdot n_b}{n_i\cdot p_r+n_b\cdot p_r}\theta(E_r<E_0)(1-z_{ri}) \\
\times\Big[\Theta(p_r-z_{ri}p_i)-1\Big],
\end{aligned}
\tag{98}
$$

is finite and

$$
L_{ib,\mathrm{compl}}^{\mathrm{F,soft,div}}(\epsilon) = \frac{-2}{\pi_\epsilon\mu^{\bar\epsilon}}\int d^{4+\bar\epsilon}p_r\delta_+(p_r^2)\frac{1}{n_i\cdot p_r}\frac{n_i\cdot n_b}{n_i\cdot p_r+n_b\cdot p_r}\theta(E_r<E_0)(1-z_{ri}). \tag{99}
$$

We find it too complicated for analytical computation. However, we introduce

$$
E_r^{(ib)} = E_r\frac{n_r\cdot n_b+n_i\cdot n_r}{n_i\cdot n_b} \tag{100}
$$

which in collinear limit is equal to E_r and goes to 0 in the soft limit. With that we define

$$
\begin{aligned}
L_{ib}^{\mathrm{F,soft,fin}} = \frac{-2}{\pi}\int d^4p_r\delta_+(p_r^2)\frac{1}{n_i\cdot p_r}\frac{n_i\cdot n_b}{n_i\cdot p_r+n_b\cdot p_r} \\
\times\Bigg[\Theta(p_r-z_{ri}p_i)\theta(E_r<E_0)\left(1-\frac{E_r}{E_i}\right) \\
-\theta(E_r^{(ib)}<E_0)\left(1-\frac{E_r^{(ib)}}{E_i}\right)\Bigg],
\end{aligned}
\tag{101}
$$

and

$$
L_{ib}^{\mathrm{F,soft,div}}(\epsilon) = \frac{-2}{\pi_\epsilon\mu^{\bar\epsilon}}\int d^{4+\bar\epsilon}p_r\delta_+(p_r^2)\frac{1}{n_i\cdot p_r}\frac{n_i\cdot n_b}{n_i\cdot p_r+n_b\cdot p_r}\theta(E_r^{(ib)}<E_0)\left(1-\frac{E_r^{(ib)}}{E_i}\right), \tag{102}
$$

which is simpler then the previous one.

3.3.3 Soft-collinear terms

For soft-collinear terms with analogy we have

$$
L_i^{\mathrm{F,soco,fin}} = \frac{-2C_i}{\pi}\int d^4p_r\delta_+(p_r^2)\mathcal{S}^{\mathrm{F,soco}}(p_r;p_i)\Big[\Theta(p_r-z_{ri}p_i)-1\Big], \tag{103}
$$

$$
L_{ir}^{\mathrm{F,coll,div}}(\epsilon) = \frac{-2C_i}{\pi_\epsilon\mu^{\bar\epsilon}}\int d^{4+\bar\epsilon}p_r\delta_+(p_r^2)\mathcal{S}^{\mathrm{F,soco}}(p_r;p_i), \tag{104}
$$

and

$$
\mathcal{S}^{\mathrm{F,soco}}(p_r;p_i) = \theta(E_r<E_0)\theta(n_r\cdot n_i<2\zeta_0)(1-z_{ri})\frac{1}{p_i\cdot p_r}\frac{1}{z_{ri}}. \tag{105}
$$

3.4 Initial-state terms

The other terms in equation (77) involve the arguments $\left(Q - p_r; \{p_i\}_n^{\not f}\right)$. In these terms, the radiator is one of the initial-state partons $(k_{\rm in}, k_{\overline{\rm in}})$. We will focus on the terms corresponding to $k_{\rm in}$, as the terms related to $k_{\overline{\rm in}}$ can be obtained by substituting non-bar variables with the bar ones. The formulas are similar to those for the final-state terms, but we prefer to express them using the variables $\overline{x}_r = \frac{2P\cdot p_r}{S}$ and $x_r = \frac{2\overline{P}\cdot p_r}{S}$. Then, we write

$$\mathcal{R}^{\rm I,coll}_{k_{\rm in}r}\mathcal{A}^{\rm I,coll}_{k_{\rm in}r} = \frac{4\pi\alpha_s}{\mu^{\bar\epsilon}}\theta(\overline{x}_r < \xi_0 x_r)\frac{-2}{S\overline{x}_r x}\mathcal{Q}_{k_{\rm in}r}(-x_r/x)|\mathcal{M}_{k_{\rm in}r}|^2, \tag{106}$$

$$\mathcal{R}^{\rm I,soft}_{k_{\rm in}}\mathcal{A}^{\rm I,soft}_{k_{\rm in}} = -\frac{4\pi\alpha_s}{\mu^{\bar\epsilon}}\theta(E_r < E_0)\frac{2}{n_{k_{\rm in}}\cdot p_r}\sum_b \frac{n_{k_{\rm in}}\cdot n_b}{n_{k_{\rm in}}\cdot p_r + n_b\cdot p_r}(\mathcal{M})^2_{\text{color}(k_{\rm in},b)}, \tag{107}$$

$$\mathcal{R}^{\rm I,soco}_{k_{\rm in}}\mathcal{A}^{\rm I,soco}_{k_{\rm in}} = -\frac{4\pi\alpha_s}{\mu^{\bar\epsilon}}\theta(E_r < E_0)\theta(\overline{x}_r < \xi_0 x_r)\frac{4C_{k_{\rm in}}}{Sx_r\overline{x}_r}|\mathcal{M}|^2. \tag{108}$$

For the space-like $k_{\rm in}$, the collinear singularity appears only for gluonic radiation, so there is no need for a subtraction term involving quark or antiquark radiation. In the case of $k_{\overline{\rm in}}$, all collinear terms are already included.

Let us introduce the splitting function $\mathcal{Q}_{k_{\rm in}r}(\zeta)$, which can be obtained by taking the collinear limit of the off-shell squared matrix element as shown in equation (47)

$$|\overline{\mathcal{M}}^\star|^2\left(x, k_T, \overline{x}; r, \{p_i\}_n\right) \xrightarrow{r\to x_r P} \frac{-1}{P\cdot r}\frac{2N_c}{\zeta(1+\zeta)^2}|\overline{\mathcal{M}}^\star|^2\left((x-x_r), k_T, \overline{x}; r, \{p_i\}_n\right) \tag{109}$$

with $\zeta = -x_r/x$. The splitting functions $\mathcal{Q}$ are formulated to be related to matrix elements with all momenta outgoing, and are negative for the initial-state case when the energy ratio is negative. The momentum P refers to an initial state, but is defined having positive energy, and the overall minus sign in equation (109) corrects for this.

$$\mathcal{Q}_{k_{\rm in}r}(\zeta) = \frac{2C_g}{\zeta(1+\zeta)^2} \quad\Leftrightarrow\quad \mathcal{P}_{k_{\rm in}r}(z) = -z\mathcal{Q}_{k_{\rm in}r}(z-1) = \frac{2C_g}{z(1-z)}. \tag{110}$$

This relation will become clear in the derivation of equation (136).

To proceed, we need to connect equations (81) and (82) with (106), (107), and (108). We will explicitly write the Θ function as defined in equation (83). It is important to note that we will focus only on the $k_{\rm in}$ terms here, as the corresponding $k_{\overline{\rm in}}$ terms can be obtained by simply exchanging the bar and non-bar variables.

3.4.1 Soft terms

Firstly we introduce the following abbreviation

$$[dV(p_r,\epsilon)] = \frac{-2}{\pi_\epsilon \mu^{\bar{\epsilon}}} d^{4+\bar{\epsilon}} p_r \delta_+(p_r^2) \frac{1}{E_r^2 n_{k_{\rm in}} \cdot n_r}. \tag{111}$$

We need to evaluate the following integral

$$\begin{aligned} L^{\rm I,soft}_{k_{\rm in}b}(\epsilon) = \int [dV(p_r,\epsilon)] \frac{n_{k_{\rm in}} \cdot n_b}{n_r \cdot n_{k_{\rm in}} + n_r \cdot n_b} \\ \times \theta(x_r < 1-x)\theta(\overline{x}_r < 1-\overline{x})\theta(E_r < E_0). \end{aligned} \tag{112}$$

For any initial-state soft or collinear limit p_{rT} vanishes. Hence, a single subtraction is sufficient. We write

$$\begin{aligned} L^{\rm I,soft,fin,1}_{k_{\rm in}b} &= \int [dV(p_r,0)] \frac{n_{k_{\rm in}} \cdot n_b}{n_r \cdot n_{k_{\rm in}} + n_r \cdot n_b} \theta(x_r < 1-x) \\ &\quad \times \Big[\theta(\overline{x}_r < 1-\overline{x})\theta(E_r < E_0) - \theta\big(E_r^{(k_{\rm in}b)} < E_0\big)\Big], \end{aligned} \tag{113}$$

$$L^{\rm I,soft,div,1}_{k_{\rm in}b}(\epsilon) = \int [dV(p_r,\epsilon)] \frac{n_{kin} \cdot n_b}{n_r \cdot n_{k_{\rm in}} + n_r \cdot n_b} \theta(x_r < 1-x)\theta\big(E_r^{(k_{\rm in}b)} < E_0\big), \tag{114}$$

with $E_r^{(k_{\rm in}b)}$ defined as in equation (100). The superscript 1 indicates that this is not the final result, as $L^{\rm I,soft,div,1}_{k_{\rm in}b}(\epsilon)$ is complicated by its dependence on n_b and the presence of $\theta(x_r < 1-x)$. Therefore, we proceed with the following decomposition

$$L^{\rm I,soft,div,1}_{k_{\rm in}b}(\epsilon) = L^{\rm I,soft,div}_{k_{\rm in}b}(\epsilon) + L^{\rm I,soft,div,2}_{k_{\rm in}b}(\epsilon) + L^{\rm I,soft,fin,2}_{k_{\rm in}b}(\epsilon) + \mathcal{O}(\epsilon), \tag{115}$$

where

$$\begin{aligned} L^{\rm I,soft,fin,2}_{k_{\rm in}b} &= \int [dV(p_r,0)] \Big[\theta(1-x < E_r/E)\theta(E_r < E_0) \\ &\quad - \frac{n_{k_{\rm in}} \cdot n_b}{n_r \cdot n_{k_{\rm in}} + n_r \cdot n_b} \theta(1-x < x_r)\theta\big(E_r^{(k_{\rm in}b)} < E_0\big)\Big], \end{aligned} \tag{116}$$

$$L^{\rm I,soft,div}_{k_{\rm in}b}(\epsilon) = \int [dV(p_r,\epsilon)] \frac{n_{kin} \cdot n_b}{n_r \cdot n_{k_{\rm in}} + n_r \cdot n_b} \theta\big(E_r^{(k_{\rm in}b)} < E_0\big), \tag{117}$$

$$L^{\rm I,soft,div,2}_{k_{\rm in}b}(\epsilon) = -\int [dV(p_r,\epsilon)] \theta(1-x < E_r/E)\theta(\overline{x}_r < 1-\overline{x})\theta(E_r < E_0). \tag{118}$$

Now, thanks to the θ functions in $L^{\rm I,soft,fin,2}_{k_{\rm in}b}$, p_r is prevented from becoming soft while still allowing it to become identical to the initial-state $k_{\rm in}$. We also observe that $L^{\rm I,soft,div,2}_{k_{\rm in}b}(\epsilon)$ is independent of the spectator b, which is relevant for the entire soft and soft-collinear terms. Thus, the complete finite soft contribution is

$$L^{\mathrm{I,soft,fin}}_{k_{\mathrm{in}}b} = L^{\mathrm{I,soft,fin,1}}_{k_{\mathrm{in}}b} + L^{\mathrm{I,soft,fin,2}}_{k_{\mathrm{in}}b}. \tag{119}$$

3.4.2 Soft-collinear terms

We start with the abbreviation

$$[dW(p_r,\epsilon)] = \frac{-2C_{k_{\mathrm{in}}}}{\pi_\epsilon \mu^{\bar{\epsilon}}}\frac{2}{S} d^{4+\bar{\epsilon}} p_r \delta_+(p_r^2)\frac{\theta(E_r<E_0)\theta(\overline{x}_r<\xi_0 x_r)}{x_r\overline{x}_r}. \tag{120}$$

Our objective is to calculate

$$L^{\mathrm{I,soco}}_{k_{\mathrm{in}}}(\epsilon) = \int [dW(p_r,\epsilon)]\theta(x_r<1-x)\theta(\overline{x}_r<1-\overline{x}), \tag{121}$$

and we split it into

$$L^{\mathrm{I,soco,fin,1}}_{k_{\mathrm{in}}} = \int [dW(p_r,0)]\theta(x_r<1-x)\big[\theta(\overline{x}_r<1-\overline{x})-1\big], \tag{122}$$

$$\begin{aligned} L^{\mathrm{I,soco,div,1}}_{k_{\mathrm{in}}}(\epsilon) &= \int [dW(p_r,\epsilon)]\theta(x_r<1-x) \\ &= L^{\mathrm{I,soco,div}}_{k_{\mathrm{in}}}(\epsilon) + L^{\mathrm{I,soco,div,2}}_{k_{\mathrm{in}}}(\epsilon) + L^{\mathrm{I,soco,fin,2}}_{k_{\mathrm{in}}} + \mathcal{O}(\epsilon), \end{aligned} \tag{123}$$

with

$$L^{\mathrm{I,soco,fin,2}}_{k_{\mathrm{in}}} = \int [dW(p_r,0)]\big[\theta(1-x<E_r/E)-\theta(1-x<x_r)\big], \tag{124}$$

$$L^{\mathrm{I,soco,div}}_{k_{\mathrm{in}}}(\epsilon) = \int [dW(p_r,\epsilon)], \tag{125}$$

$$L^{\mathrm{I,soco,div,2}}_{k_{\mathrm{in}}}(\epsilon) = -\int [dW(p_r,\epsilon)]\theta(1-x<E_r/E). \tag{126}$$

The exact calculations show that, up to the color factor, the $1/\epsilon$ poles from $L^{\mathrm{I,soco,div,2}}_{k_{\mathrm{in}}}(\epsilon)$ and $L^{\mathrm{I,soft,div,2}}_{k_{\mathrm{in}}}(\epsilon)$ are identical. By applying color conservation to the sum over spectators, these poles cancel each other out. Collecting all finite parts, we write

$$L^{\mathrm{I,soco,fin}}_{k_{\mathrm{in}}} = L^{\mathrm{I,soco,fin,1}}_{k_{\mathrm{in}}} + L^{\mathrm{I,soco,fin,2}}_{k_{\mathrm{in}}} + \big[L^{\mathrm{I,soco,div,2}}_{k_{\mathrm{in}}}(\epsilon) - C_{k_{\mathrm{in}}}L^{\mathrm{I,soft,div,2}}_{k_{\mathrm{in}}}(\epsilon)\big]_{\epsilon\to 0}. \tag{127}$$

3.4.3 Collinear terms

The variables x and $\overline{x}$ from equation (106) are considered before shifting the initial-state variables. Since we are addressing the k_{in} terms, the transformation $Q \to Q + x_r P + p_{rT}$ is applied, which necessitates the shift $x \to x + x_r$ to obtain the desired result

$$L^{\mathrm{I,coll}}_{k_{\mathrm{in}}r} = \frac{1}{\pi_\epsilon \mu^{\bar\epsilon}}\frac{2}{S}\int d^{4+\bar\epsilon}p_r\delta_+(p_r^2)\mathcal{S}^{\mathrm{I,coll}}_{k_{\mathrm{in}}r}(x_r,\overline{x}_r)\ell_{k_{\mathrm{in}}}(x+x_r)\theta(x_r<1-x)\theta(\overline{x}_r<1-\overline{x}), \tag{128}$$

with

$$\mathcal{S}^{\mathrm{I,coll}}_{k_{\mathrm{in}}r}(x_r,\overline{x}_r) = \theta(\overline{x}_r<\xi_0 x_r)\frac{-1}{\overline{x}_r(x+x_r)}\mathcal{Q}_{k_{\mathrm{in}}r}\left(\frac{-x_r}{x+x_r}\right), \tag{129}$$

and

$$\ell_{k_{\mathrm{in}}}(y) = \frac{\mathcal{L}\big(yP+\overline{x}\overline{P}+k_T;\{p_i\}_n\big)}{\mathcal{L}\big(xP+\overline{x}\overline{P}+k_T;\{p_i\}_n\big)}. \tag{130}$$

Again, the $k_{\overline{\mathrm{in}}}$ terms are completely analogous, and we do not make any assumptions about the splitting $\mathcal{Q}_{k_{\mathrm{in}}}$. We rewrite equation (128) using the Sudakov decomposition of the p_r integral as follows

$$L^{\mathrm{I,coll}}_{k_{\mathrm{in}}r} = \frac{1}{\pi_\epsilon \mu^{\bar\epsilon}}\int_0^{1-x}dx_r\int_0^{1-\overline{x}}d\overline{x}_r\mathcal{S}^{\mathrm{I,coll,r}}_{k_{\mathrm{in}}}(x_r,\overline{x}_r)\int d^{2+\bar\epsilon}p_{rT}\delta_+\big(Sx_r\overline{x}_r-|p_{rT}|^2\big)\ell_{k_{\mathrm{in}}}(x+x_r). \tag{131}$$

We see that

$$\int d^{2+\bar\epsilon}p_r\delta_+\big(Sx_r\overline{x}_r-|p_{rT}|^2\big) = \pi_\epsilon(S\overline{x}_r x_r)^{\bar\epsilon/2}, \tag{132}$$

with that we define

$$\begin{aligned} L^{\mathrm{I,coll,fin}}_{k_{\mathrm{in}}r} =& \int_0^1 dx_r\int_0^1 d\overline{x}_r\mathcal{S}^{\mathrm{I,coll}}_{k_{\mathrm{in}}r}(x_r,\overline{x}_r)\theta(x_r<1-x)\\ &\times\left[\int\frac{d^2p_{rT}}{\pi}\delta_+\big(Sx_r\overline{x}_r-|p_{rT}|^2\big)\ell_{k_{\mathrm{in}}}(x+x_r)\theta(\overline{x}_r<1-\overline{x})-\ell_{k_{\mathrm{in}}}(x+x_r)\right], \end{aligned} \tag{133}$$

and

$$\begin{aligned} L^{\mathrm{I,coll,div}}_{k_{\mathrm{in}}r} =& \left(\frac{S}{\mu^2}\right)^{\bar\epsilon/2}\int_0^1 dx_r x_r^{\bar\epsilon/2}\int_0^1 d\overline{x}_r\overline{x}_r^{\bar\epsilon/2}\mathcal{S}^{\mathrm{I,coll}}_{k_{\mathrm{in}}r}(x_r,\overline{x}_r)\ell_{k_{\mathrm{in}}}(x+x_r)\theta(x_r<1-x)\\ =&\frac{2}{\bar\epsilon}\left(\frac{Sx^2\xi_0}{\mu^2}\right)^{\bar\epsilon/2}\int_0^1\frac{dx_r}{x}\left(\frac{x_r}{x}\right)^{\bar\epsilon}\frac{-x}{x+x_r}\mathcal{Q}_{k_{\mathrm{in}}r}\left(\frac{-x_r}{x+x_r}\right)\ell_{k_{\mathrm{in}}}(x+x_r)\theta(x_r<1-x). \end{aligned} \tag{134}$$

At this point is reasonable to perform the following variable substitution

$$z = \frac{x}{x + x_r} \quad \Leftrightarrow \quad x_r = \frac{1-z}{z} x, \quad dx_r = x \frac{dz}{z^2}, \tag{135}$$

and with equation (110) we obtain

$$L^{\mathrm{I,coll,div}}_{k_{\mathrm{in}} r} = \frac{2}{\bar{\epsilon}} \left(\frac{S x^2 \xi_0}{\mu^2} \right)^{\bar{\epsilon}/2} \int_0^1 dz \left(\frac{1-z}{z} \right)^{\bar{\epsilon}} \mathcal{P}_{k_{\mathrm{in}} r}(z) \frac{\ell_{k_{\mathrm{in}}}(x/z)}{z^2} \theta(z > x). \tag{136}$$

We see a factor $1/z^2$ here instead of the usual $1/z$ that typically appears in this type of expression. This arises due to the flux factor included in the $\mathcal{L}$ function. We also realize that the splitting function contains a singular term $2C_{k_{\mathrm{in}} r}/(1-z)$. For the $k_{\overline{\mathrm{in}}}$ terms with the radiated quark or antiquark, this is not an issue, and we simply set $C_{k_{\overline{\mathrm{in}}} r} = 0$. We isolate the singularity as follows

$$\begin{aligned}
L^{\mathrm{I,coll,div}}_{k_{\mathrm{in}} r} &= \frac{2}{\bar{\epsilon}} \left(\frac{S x^2 \xi_0}{\mu^2} \right)^{\bar{\epsilon}/2} \left\{ \frac{2C_{k_{\mathrm{in}} r}}{\bar{\epsilon}} + \int_0^1 dz (1-z)^{\bar{\epsilon}} \left[\mathcal{P}_{k_{\mathrm{in}} r}(z) z^{-\bar{\epsilon}} \frac{\ell_{k_{\mathrm{in}}}(x/z)}{z^2} \theta(z > x) - \frac{2C_{k_{\mathrm{in}} r}}{1-z} \right] \right\} \\
&= \left(\frac{S x^2 \xi_0}{\mu^2} \right)^{\bar{\epsilon}/2} \left\{ \frac{4C_{k_{\mathrm{in}} r}}{\bar{\epsilon}^2} + \frac{2}{\bar{\epsilon}} \int_0^1 dz \big[1 + \bar{\epsilon} \ln(1-z) \big] \mathcal{P}^{\mathrm{reg}}_{k_{\mathrm{in}} r}(z) \frac{\ell_{k_{\mathrm{in}}}(x/z)}{z^2} \theta(z > x) \right. \\
&\qquad \left. - 2 \int_0^1 dz \ln(z) \mathcal{P}_{k_{\mathrm{in}} r}(z) \frac{\ell_{k_{\mathrm{in}}}(x/z)}{z^2} \theta(z > x) + \mathcal{O}(\bar{\epsilon}) \right\},
\end{aligned} \tag{137}$$

where

$$\mathcal{P}^{\mathrm{reg}}_{k_{\mathrm{in}} r}(z) = \mathcal{P}_{k_{\mathrm{in}} r}(z) - \frac{2C_{k_{\mathrm{in}} r}}{1-z} + \frac{2C_{k_{\mathrm{in}} r}}{[1-z]_+} \tag{138}$$

and the plus-distribution is defined as

$$\int_0^1 dz f(z) \frac{1}{[1-z]_+} g(z) = \int_0^1 dz f(z) \frac{1}{1-z} \big[g(z) - g(1) \big]. \tag{139}$$

3.5 Integrated subtraction terms

As we mentioned before, we present here the results of the integrals from $L^{X,Y,\mathrm{div}}_{Z}(\epsilon)$. The detailed calculations are provided in Appendix F of [2]. In Appendix G of [2], we show the manipulations of $L^{X,Y,\mathrm{fin}}_{Z}$ to bring it to a suitable form.

We find the final-state collinear terms to be

$$L^{\mathrm{F,coll,div}}_{ir}(\epsilon) = \left(\frac{\mu^2}{4E_i^2} \right)^{\epsilon} \left[-\frac{1}{\epsilon} + \ln(\zeta_0) + \epsilon \left(\mathrm{Li}_2(\zeta_0) - \frac{1}{2} \ln^2(\zeta_0) \right) + \mathcal{O}(\epsilon^2) \right] I_{ir}(\epsilon), \tag{140}$$

with

$$I_{ir}(\epsilon) = \int_0^{1/2} dx x^{-2\epsilon} \mathcal{P}_{ir}(1-x), \tag{141}$$

and for given i and r

$$I_{gg}(\epsilon) = C_g \left[-\frac{1}{\epsilon} - \frac{11}{6} - \epsilon \left(\frac{11\ln 2}{2} - \frac{\pi^2}{3} + \frac{137}{36} \right) + \mathcal{O}(\epsilon^2) \right], \tag{142}$$

$$I_{qg}(\epsilon) = C_q \left[-\frac{1}{\epsilon} - 2\ln 2 - \frac{7}{8} - \epsilon \left(2\ln^2 2 + \frac{7\ln 2}{4} + 2 \right) + \mathcal{O}(\epsilon^2) \right], \tag{143}$$

$$I_{gq}(\epsilon) = C_q \left[2\ln 2 - \frac{5}{8} + \epsilon \left(2\ln^2 2 - \frac{5\ln 2}{4} + \frac{\pi^2}{3} - \frac{3}{2} \right) + \mathcal{O}(\epsilon^2) \right], \tag{144}$$

$$I_{qq}(\epsilon) = T_R \left[\frac{1}{3} + \epsilon \left(\frac{2\ln 2}{2} + \frac{23}{36} \right) + \mathcal{O}(\epsilon^2) \right]. \tag{145}$$

For the soft and soft-collinear terms as well finial-state as initial-state, we have

$$L_{ib}^{\mathrm{F,soft,div}}(\epsilon) = \left(\frac{\mu^2}{2E_0^2 n_i \cdot n_b} \right)^{\epsilon} \left[-\frac{1}{\epsilon^2} - \frac{1}{\epsilon} \frac{2E_0}{E_i} + \frac{\pi^2}{6} - \frac{4E_0}{E_i} + \mathcal{O}(\epsilon) \right], \tag{146}$$

$$\begin{aligned} L_{i}^{\mathrm{F,soco,div}}(\epsilon) = C_i \left(\frac{\mu^2}{4E_0^2} \right)^{\epsilon} \Bigg[& -\frac{1}{\epsilon^2} + \frac{1}{\epsilon} \left(\ln(\zeta_0) - \frac{2E_0}{E_i} \right) \\ & + \frac{2E_0}{E_i} \ln(\zeta_0) + \mathrm{Li}_2(\zeta_0) - \frac{1}{2}\ln^2(\zeta_0) - \frac{4E_0}{E_i} + \mathcal{O}(\epsilon) \Bigg], \end{aligned} \tag{147}$$

$$L_{k_{\mathrm{in}}b}^{\mathrm{I,soft,div}}(\epsilon) = \left(\frac{\mu^2}{2E_0^2 n_{k_{\mathrm{in}}} \cdot n_b} \right)^{\epsilon} \left[-\frac{1}{\epsilon^2} + \frac{\pi^2}{6} + \mathcal{O}(\epsilon) \right], \tag{148}$$

$$L_{k_{\mathrm{in}}}^{\mathrm{I,soco,div}}(\epsilon) = C_{k_{\mathrm{in}}} \left(\frac{\mu^2 E}{4E_0^2 \overline{E} \xi_0} \right)^{\epsilon} \left[-\frac{1}{\epsilon^2} - 2\mathrm{Li}_2 \left(\frac{-\overline{E}\xi_0}{E} \right) + \mathcal{O}(\epsilon) \right]. \tag{149}$$

And finally the initial-state collinear term is

$$\begin{aligned} L_{k_{\mathrm{in}}r}^{\mathrm{I,coll,div}}(\epsilon) = \left(\frac{\mu^2 E}{4E^2 x^2 \overline{E} \xi_0} \right)^{\epsilon} \Bigg\{ & \frac{C_{k_{\mathrm{in}}r}}{\epsilon^2} - \frac{1}{\epsilon} \int_0^1 dz \mathcal{P}_{k_{\mathrm{in}}r}^{\mathrm{reg}}(z) \frac{\ell_{k_{\mathrm{in}}}(x/z)}{z^2} \theta(z > x) \\ & + 2 \int_0^1 dz \left[\ln(1-z) \mathcal{P}_{k_{\mathrm{in}}r}^{\mathrm{reg}}(z) - \ln(z) \mathcal{P}_{k_{\mathrm{in}}r}(z) - \frac{1}{2} \mathcal{P}_{k_{\mathrm{in}}r}^{(1)}(z) \right] \\ & \times \frac{\ell_{k_{\mathrm{in}}}(x/z)}{z^2} \theta(z > x) + \mathcal{O}(\overline{\epsilon}) \Bigg\}, \end{aligned} \tag{150}$$

with $\mathcal{P}^{\rm reg}_{k_{\rm in}r}(z)$ defined in equation (138) and $\ell_{k_{\rm in}}$ defined in equation (130). The splittings are expanded in ϵ, where $\mathcal{P}^{(1)}_{k_{\rm in}r}(z)$ represents the coefficient of $\mathcal{O}(\epsilon)$. To obtain the result for $k_{\overline{\rm in}}$ terms, the following substitutions should be made $x \leftrightarrow \overline{x}$, $P \leftrightarrow \overline{P}$, and $E \leftrightarrow \overline{E}$. For non-gluonic radiation, we set $C_{k_{\overline{\rm in}}r} = 0$ in the formula above.

Equation (150) pertains to the integrals of the subtraction term labeled r in equation (77). In cases where r refers to a quark or antiquark pair, the soft and soft-collinear (soco) terms vanish. We will now focus on the terms for $r \to g$. We introduce the following abbreviation

$$(m)^2_{a,b} = (\mathcal{M})^2_{{\rm color}(a,b)}/|\mathcal{M}|^2. \tag{151}$$

With color conservation we can easily see that

$$L^{\rm F,coll,div}_{ig}(\epsilon) + L^{\rm F,soco,div}_{i}(\epsilon) + \sum_{b\neq i} L^{\rm F,soft,div}_{ib}(\epsilon)(m)^2_{i,b} = {\rm Soft}_i(\epsilon) + \frac{\gamma^{(g)}_i}{\epsilon} + \mathcal{O}(\epsilon^0), \tag{152}$$

and

$$\begin{aligned} L^{\rm I,coll,div}_{k_{\rm in}g}(\epsilon) + L^{\rm I,soco,div}_{k_{\rm in}}(\epsilon) + \sum_{b\neq k_{\rm in}} L^{\rm I,soft,div}_{k_{\rm in}b}(\epsilon)(m)^2_{k_{\rm in},b} = \\ {\rm Soft}_{k_{\rm in}}(\epsilon) - \frac{1}{\epsilon}\int_0^1 dz \mathcal{P}^{\rm reg}_{k_{\rm in}g}(z)\frac{\ell_{k_{\rm in}}(x/z)}{z^2}\theta(z>x) + \mathcal{O}(\epsilon^0), \end{aligned} \tag{153}$$

with

$${\rm Soft}_a(\epsilon) = \frac{C_a}{\epsilon^2} - \frac{C_a}{\epsilon}{\rm ln}\left(\frac{\mu^2}{E_a^2}\right) + \frac{1}{\epsilon}\sum_{b\neq a}{\rm ln}\left(\frac{1}{2n_a\cdot n_b}\right)(m)^2_{a,b}, \tag{154}$$

and

$$\gamma^{(g)}_i = C_i \times \begin{cases} 11/6 & \text{if } i \text{ is a gluon} \\ 2{\rm ln}2 + 7/8 & \text{if } i \text{ is a quark or antiquark} \end{cases} \tag{155}$$

We immediately observe that the result is independent of the parameters E_0, ζ_0, ξ_0. Upon summing over a in equation (154) and applying color conservation, we recover the well-known universal formula for the soft and soft-collinear divergences [5]. With

$$\sum_{a,b;b\neq a}{\rm ln}\left(\frac{\mu^2}{2p_a\cdot p_b}\right)(m)^2_{a,b} = -2\sum_a C_a{\rm ln}\left(\frac{\mu}{E_a}\right) + \sum_{a,b;b\neq a}{\rm ln}\left(\frac{1}{2n_a\cdot n_b}\right)(m)^2_{a,b} \tag{156}$$

we obtain

$$
\begin{aligned}
\sum_a \mathrm{Soft}_a(\epsilon) &= \sum_a \frac{C_a}{\epsilon^2} + \frac{1}{\epsilon} \sum_{a,b;b\neq a} \ln\left(\frac{\mu^2}{2p_a \cdot p_b}\right)(m)^2_{a,b} \\
&= \sum_{a,b} \left(\frac{\mu^2}{2p_a \cdot p_b}\right)^{\epsilon} \frac{(m)^2_{a,b}}{\epsilon^2} + \mathcal{O}(\epsilon^0), \quad \text{with} \quad (m)^2_{a,a} = C_a.
\end{aligned}
\tag{157}
$$

The remaining collinear term in equation (153) is as expected for the on-shell $k_{\overline{\mathrm{in}}}$ side, and anticipated to cancel against a collinear counterterm prescribed by factorization. For the space-like k_{in} side, the term in equation (153) is identical to the divergent term in equation (58) and the splitting function is given in equation (110). Thus, we obtained the result of equation (58), without any approximations.

The form of divergent contribution that we addressed above appeared because of the concentration on a given radiative process. For cancellation of all divergences, all sub processes relevant for a given jet process, have to be included. Nevertheless, the soft divergent contribution of equation (157) will match the universal expression for the soft virtual contribution. It happen, because the Born process, which form the NLO real and virtual contribution, can be obtained by removal a soft gluon from a radiative process. In order to include all collinear divergences in the processes with at least one gluon involved is obtained by adding the contribution of equation (144), when the radiative quark is collinear to a gluon to the equation (155). Thus, we have

$$
\gamma_g = \gamma_g^{(g)} = \frac{11}{6} C_g, \quad \gamma_q = \gamma_q^{(g)} + \gamma_g^{(g)} = \frac{3}{2} C_q, \tag{158}
$$

and we will see that γ_q matches the formula for the virtual contribution.

In case for getting the fermion-loop term for γ_g included we need to go beyond the interest of the single Born process. We realize that considering radiation with final-state collinear quark-antiquark pair, there is a divergence $-T_R/(3\epsilon)$ as in equation (145) to the Born process with that pair replaced with a gluon. The same is for initial-state and final-state quarks. Then the revision of the roles of quark and antiquark produce a factor of 2 and a factor which counts the number of quark families. Hence, we find

$$
\gamma_g = \frac{11}{6} C_g - \frac{2}{3} T_R n_f. \tag{159}
$$

4 Virtual contribution

Computation of the virtual contribution demands an analogous approach to the Born level, as both involve the same phase space. Thus, to calculate the virtual contribution, we use equations (28), (33), and (35), with $|\overline{\mathcal{M}}^*|^2$ replaced with

$$|\overline{\mathcal{M}}^*|^2 \longrightarrow 2Re\{\overline{\mathcal{M}}^{*\dagger}\overline{\mathcal{M}}^{*\text{one-loop}}\} \tag{160}$$

As argued in [14, 16], one-loop off-shell amplitudes can be computed using existing on-shell formulas within the auxiliary parton method. The generalization and broader application of this method were emphasized in [28]. The introduction of auxiliary partons allows for the numerical computation of certain processes, paving the way for automation. Additionally, this method is well-established at the tree level, facilitating the accurate computation of real contributions to NLO cross sections within the auxiliary parton framework. Therefore, we adopt the approach described in [14, 16, 28] to determine the virtual contribution at NLO.

4.1 Auxiliary parton method at amplitude level

The auxiliary parton method at the amplitude level was studied in [14, 16], and the role of this subsection is to summarize it. In those works, the notation of the spinor helicity method was utilized, which we have reserved for appendix A. We define the amplitude with a space-like gluon as follows

$$\frac{1}{\Lambda}\mathcal{M}(A, B, ...) \xrightarrow{\Lambda\to\infty} \mathcal{M}^*(g^*, ...), \tag{161}$$

where this amplitude is gauge invariant. Following [14], we omit a factor $i|k_T|$ from the amplitudes, with the remark that this factor must be included in the final expression to ensure the proper on-shell limit. Additionally, we set g_S to be equal to 1. It is worth noting that in the end, we need to include a factor of g_S^{n-2} in the tree-level matrix element with n partons involved and a factor of g_S^n in the one-loop expressions. The letters A and B denote an auxiliary quark-antiquark pair or an auxiliary gluon pair. In equation (161), we omitted the color dependence, and we will revisit this at the end of this subsection.

We decompose the transverse momentum k_T with the help of arbitrary momentum q^μ in construction of the polarization vectors.

$$k_T^\mu = -\frac{\kappa^*}{\sqrt{2}}e_+^\mu - \frac{\kappa}{\sqrt{2}}e_-^\mu \tag{162}$$

with

$$\kappa^* = \frac{\langle p|\not{k}|q]}{[pq]}, \quad \kappa = \frac{\langle q|\not{k}|p]}{\langle qp\rangle} \tag{163}$$

and polarization vectors are defined

$$e_+^\mu = \frac{\langle q|\gamma^\mu|p]}{\sqrt{2}\langle qp\rangle}, \quad e_-^\mu = \frac{\langle p|\gamma^\mu|q]}{\sqrt{2}[pq]}. \tag{164}$$

Note that κ and κ^* do not depend on the momentum fraction x and also do not depend on the arbitrary momentum q^μ [26]. Thus, we obtain

$$\kappa\kappa^* = -k_T^2. \tag{165}$$

The Weyl spinors for momenta of the auxiliary partons introduced in equation (18), here A stays for k_1 and B for k_2, we write

$$\begin{aligned} |A\rangle &= \sqrt{\Lambda}|p\rangle - \frac{\beta\kappa^*}{\sqrt{\Lambda}\langle qp\rangle}|q\rangle \qquad |A] = \sqrt{\Lambda}|p] - \frac{\beta\kappa}{\sqrt{\Lambda}[pq]}|q], \\ |B\rangle &= \sqrt{\Lambda-1}|p\rangle - \frac{\beta\kappa^*}{\sqrt{\Lambda}\langle qp\rangle}|q\rangle \quad |B] = -\sqrt{\Lambda-1}|p] - \frac{\beta\kappa}{\sqrt{\Lambda}[pq]}|q], \end{aligned} \tag{166}$$

with β defined in equation (20).

In order to perform the limit from equation (161), one should practically apply the substitution given in equation (166) to the given on-shell amplitude in the following order:

1.

$$\langle AB\rangle \to -\kappa^*, \quad [AB] \to -\kappa, \quad k_A^\mu + k_B^\mu \to k^\mu \tag{167}$$

and also

$$s_{AB} = (k_A + k_B)^2 \to k^2 = -\kappa\kappa^*, \quad t_{ABi} = (k_A + k_B + p_i)^2 \to (k + p_i)^2 = s_{ki}. \tag{168}$$

2.

$$k_A^\mu \to \Lambda p^\mu, \quad k_B^\mu \to -\Lambda p^\mu, \tag{169}$$

$$s_{Ai} \to \Lambda s_{pi}, \quad s_{Bi} \to -\Lambda s_{pi}. \tag{170}$$

3.

$$|A\rangle \to \sqrt{\Lambda}|p\rangle, \quad |A] \to \sqrt{\Lambda}|p], \quad |B\rangle \to \sqrt{\Lambda}|p\rangle, \quad |B] \to -\sqrt{\Lambda}|p]. \tag{171}$$

As the result shows that the whole expression behaves with the leading power of Λ^1, we will obtain the correct formula for an off-shell amplitude. If there is a power of Λ greater than 1 implying cancellations between different terms with this behaviour, then we need to include all terms from equation (166).

We will adhere to the convention of the so-called color decomposition, extensively detailed in papers by Z. Bern et al., such as [29]. For instance, for a pure gluonic amplitude at tree-level, we have

$$\mathcal{M}_n^{\text{tree}}(1^*, ..., n) = \sum_{\sigma \in S_n/Z_n} Tr\left(T^{a_{\sigma(1)}}...T^{a_{\sigma(n)}}\right) \mathcal{A}_n^{\text{tree}}\left(\sigma(1^*), ..., \sigma(n)\right). \tag{172}$$

The $\mathcal{M}$ amplitude depends on momenta, colors, helicities of n gluons, while on the right hand site the dependence on color is decomposed within a trace of T color generators, satisfying color algebra

$$Tr\left(T^a T^b\right) = \delta^{ab}, \quad [T^a, T^b] = \sqrt{2} i f^{abc} T^c, \quad f^{123} = 1. \tag{173}$$

For one-loop amplitudes the decomposition gives

$$\begin{aligned} \mathcal{M}^n(1^*, ..., n) = &\sum_{\sigma \in S_n/Z_n} N_c Tr\left(T^{a_{\sigma(1)}}...T^{a_{\sigma(n)}}\right) \mathcal{A}_1^n\left(\sigma(1^*), ..., \sigma(n)\right) \\ + &\sum_{c=2}^{[n/2+1]} \sum_{\sigma \in S_{n;c}/Z_n} Tr\left(T^{a_{\sigma(1)}}...T^{a_{\sigma(c-1)}}\right) Tr\left(T^{a_{\sigma(c)}}...T^{a_{\sigma(n)}}\right) \mathcal{A}_c^n\left(\sigma(1^*), ..., \sigma(n)\right) \end{aligned} \tag{174}$$

At one-loop, we will omit superscripts as in the tree-level case. Thus, by $\mathcal{M}$ without any scripts, we will refer to one-loop amplitudes. It is important to note that in equation (174), additional color structures appear. $S_{n;c}$ denotes the subset of S_n that takes into account the double trace structure invariance. In general, the subscript c indicates the color structure, where $c = 1$ denotes the leading color structure. Later, partial amplitudes $\mathcal{A}$ will be expressed as combinations of primitive amplitudes, which will be indicated by appropriate superscripts. It should be emphasized that gauge invariance is ensured for primitive amplitudes as well.

Originally, the color decomposition was introduced for on-shell amplitudes. While it remains valid for off-shell amplitudes, we denote the off-shell gluon by 1^*. The helicity of a gluon will be indicated by 2^h, where h refers to either + or -. For quarks or antiquarks, we denote them with the appropriate subscript.

It is worth mentioning a relation that will allow for a clear correspondence with the real contribution at NLO

$$g_S^2 c_\Gamma = \frac{g_S^2}{(4\pi)^{2-\epsilon}} \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} = \frac{a_\epsilon}{2} + \mathcal{O}(\epsilon^3), \tag{175}$$

with a_ϵ given in equation (37). Additionally, we define two functions which will be used later. They will typically accompany some amplitudes. We write

$$\Theta(s) = 2i\pi\theta(s), \quad \Upsilon(s) = i\pi \text{sgn}(s), \tag{176}$$

where s is an invariant involved in the given process.

As mentioned in appendix B of [1], the limit of equation (21) is valid within the color summation, but at the level of amplitudes before squaring and summation, the relation given by equation (161) actually involves auxiliary parton dependence, as they have different color representations. We write the amplitude with explicitly given color indices i for the final auxiliary quark and j for the initial one. We also assume n gluons involved in a process. As the quark pair within the auxiliary parton method represents an off-shell gluon, the following relation to match the degrees of freedom for quark and gluon representation of SU(3) has to be valid. Indeed, in [11], it was shown with the help of Einstein summation

$$\mathcal{M}_{\text{aux-q}}^{*a_1 a_2 \ldots a_n ji} \delta_{ij} = 0, \tag{177}$$

where the Λ limit is already performed, and the label aux-q refers only to the different color representation. We can also decompose the amplitude into partial amplitudes using

$$\mathcal{M}_{\text{aux-q}}^{*a_1 a_2 \ldots a_n ji} = 2^{n/2} \sum_{\sigma \in S_n} (T^{a_{\sigma(1)}} T^{a_{\sigma(2)}} \cdots T^{a_{\sigma(n)}})_{ji} \mathcal{A}\left(g^*, \sigma(1), \sigma(2), \cdots, \sigma(n)\right). \tag{178}$$

Thanks to the work in [28], it has been proven that for tree-level amplitudes, it does not matter whether one chooses a quark pair or a gluon pair as auxiliary partons. It is straightforward to contract with the $\sqrt{2}T^b_{ij}$ factor to obtain the exact color decomposition of the gluon amplitude with $n+1$ legs and one off-shell gluon with color index b

$$\begin{aligned} \mathcal{M}_{\text{aux-q}}^{*a_1 a_2 \ldots a_n ji} \sqrt{2} T^b_{ij} &= 2^{(n+1)/2} \sum_{\sigma \in S_n} Tr(T^b T^{a_{\sigma(1)}} T^{a_{\sigma(2)}} \cdots T^{a_{\sigma(n)}}) \mathcal{A}\left(g^*, \sigma(1), \sigma(2), \cdots, \sigma(n)\right) \\ &= \mathcal{M}^{*b a_1 a_2 \cdots a_n} \end{aligned} \tag{179}$$

This can also be done other way around

$$\mathcal{M}^{*ba_1a_2\cdots a_n}\sqrt{2}T^b_{kl} = \mathcal{M}^{*a_1a_2\cdots a_nji}_{\text{aux-q}} 2T^b_{ij}T^b_{kl} = \mathcal{M}^{*a_1a_2\cdots a_nji}_{\text{aux-q}}\left(\delta_{il}\delta_{kj} - \frac{1}{N_c}\delta_{ij}\delta_{kl}\right) = \mathcal{M}^{*a_1a_2\cdots a_nkl}_{\text{aux-q}}. \tag{180}$$

It is also worth mentioning that the partial amplitudes for auxiliary gluons that are not adjacent are different from those for auxiliary quarks, but those amplitudes vanish in the limit $\Lambda \to \infty$. We denote auxiliary gluons by the numbers $n+1$ and $n+2$, then the amplitude with $2 \cdot n!$ non-vanishing (in the Λ limit) terms is decomposed as follows

$$\begin{aligned}\mathcal{M}^{*a_1a_2\dots a_{n+2}}_{\text{aux-g}} = 2^{n/2+1}&\sum_{\sigma\in S_n} Tr(T^{a_{n+2}}T^{a_{\sigma(1)}}\cdots T^{a_{\sigma(n)}}T^{a_{n+1}})\mathcal{A}^*_{\text{aux-g}}\left(n+2,\sigma(1),\cdots,\sigma(n),n+1\right)\\ +2^{n/2+1}&\sum_{\sigma\in S_n} Tr(T^{a_{n+2}}T^{a_{n+1}}T^{a_{\sigma(1)}}\cdots T^{a_{\sigma(n)}})\mathcal{A}^*_{\text{aux-g}}\left(n+2,n+1,\sigma(1),\cdots,\sigma(n)\right).\end{aligned} \tag{181}$$

Here, we retained the aux-g label in partial amplitudes. However, exchanging the auxiliary gluons results in changing the sign of P. Thus, we write

$$\begin{aligned}\mathcal{M}^{*a_1a_2\dots a_{n+2}}_{\text{aux-g}} &= 2^{n/2+1}\sum_{\sigma\in S_n} Tr([T^{a_{n+1}},T^{a_{n+2}}]T^{a_{\sigma(1)}}\cdots T^{a_{\sigma(n)}})\mathcal{A}^*\left(g^*,\sigma(1),\cdots,\sigma(n)\right)\\ &= 2^{n/2+1}if^{a_{n+1}a_{n+2}b}\sum_{\sigma\in S_n} Tr(T^bT^{a_{\sigma(1)}}\cdots T^{a_{\sigma(n)}})\mathcal{A}^*\left(g^*,\sigma(1),\cdots,\sigma(n)\right).\end{aligned} \tag{182}$$

In conclusion we obtained that the amplitudes follow the relations given by

$$\begin{aligned}\mathcal{M}^{a_1a_2\cdots a_nji}(q_j,\overline{q}_i,1,\cdots,n) &\xrightarrow{\Lambda\to\infty} \sqrt{2}T^b_{ji}\mathcal{M}^{*ba_1\cdots a_n}(g^*,1,\cdots,n)\\ \mathcal{M}^{a_1a_2\cdots a_na_{n+1}a_{n+2}}(g_{n+1},g_{n+2},1,\cdots,n) &\xrightarrow{\Lambda\to\infty} \sqrt{2}if^{a_{n+1}a_{n+2}b}\mathcal{M}^{*ba_1\cdots a_n}(g^*,1,\cdots,n),\end{aligned} \tag{183}$$

where the notation used is more familiar to that used in [1], and the normalization of the T color generators remains the same as in that paper. These formulas can be expressed in an inverted way

$$\begin{aligned}\frac{1}{\sqrt{2}}T^b_{ij}\mathcal{M}^{a_1a_2\cdots a_nji}(q_j,\overline{q}_i,1,\cdots,n) &\xrightarrow{\Lambda\to\infty} \mathcal{M}^{*ba_1\cdots a_n}(g^*,1,\cdots,n)\\ \frac{-i}{\sqrt{2}N_c}f^{a_{n+1}a_{n+2}b}\mathcal{M}^{a_1a_2\cdots a_na_{n+1}a_{n+2}}(g_{n+1},g_{n+2},1,\cdots,n) &\xrightarrow{\Lambda\to\infty} \mathcal{M}^{*ba_1\cdots a_n}(g^*,1,\cdots,n).\end{aligned} \tag{184}$$

Both equations (183) and (184) are obtained for tree-level amplitudes, but they are still valid for tree-like, leading-color terms. Furthermore, we will see later that in our computations the contraction of the one-loop amplitudes with tree-level ones given by equation (160) results in only leading color structures contribution to the virtual contribution that we will call unfamiliar.

At tree-level, the auxiliary parton method given by equation (161) precisely defines an off-shell amplitude. However, at one-loop, this property does not hold, as the Λ limit produces $\ln\Lambda$ terms in the formula and the entire result depends on the type of auxiliary partons used. The terms with $\ln\Lambda$ constitute the virtual contribution that we call unfamiliar and it also do not have a smooth limit $k_T \to 0$ due to presence of an explicit $\ln|kT|$. We briefly mention that the other terms contribute to the familiar contribution, where no $\ln\Lambda$ is present in the formulas, and they carry similar divergences to those already known from the on-shell expressions.

Our main interest is the computation of a cross section. In [14], there is a statement that the helicity sum must be done before taking the Λ limit. For tree-level, we observed that the helicity configurations that contribute to the NLO correction are the opposite-helicity ones. Moreover, both outcomes are equal. For auxiliary quarks, we can write

$$\begin{aligned}\mathcal{V}_q &= 2Re\Big\{\mathcal{M}^{\text{tree}\dagger}(A^+_{\bar{q}}, B^-_q)\mathcal{M}(A^+_{\bar{q}}, B^-_q) + \mathcal{M}^{\text{tree}\dagger}(A^-_{\bar{q}}, B^+_q)\mathcal{M}(A^-_{\bar{q}}, B^+_q)\} \\ &= 2Re\{\mathcal{M}^{\text{tree}\dagger}(A^+_{\bar{q}}, B^-_q)\left[\mathcal{M}(A^+_{\bar{q}}, B^-_q) + \mathcal{M}(A^-_{\bar{q}}, B^+_q)\right]\Big\}\end{aligned} \tag{185}$$

where the helicity sum is indicated only for auxiliary partons. In that formula, it is considered for an anti-quark to be the initial state. Thus, one can state that the case where there is an initial state quark has to be also included. At the amplitude level, this will be done by simply exchanging the role of A and B, which at tree-level is achieved by the change of the overall sign. Then, we write

$$\begin{aligned}\mathcal{V}_q + \mathcal{V}_{\bar{q}} = 2Re\Big\{\mathcal{M}^{\text{tree}\dagger}(A^+_{\bar{q}}, B^-_q)\Big[&\mathcal{M}(A^+_{\bar{q}}, B^-_q) + \mathcal{M}(A^-_{\bar{q}}, B^+_q) \\ &- \mathcal{M}(B^+_{\bar{q}}, A^-_q) - \mathcal{M}(B^-_{\bar{q}}, A^+_q)\Big]\Big\}\end{aligned} \tag{186}$$

For auxiliary gluons the same symmetrization procedure should be applied for A and B. Then, we write the same combination with a factor of $1/2$

$$\mathcal{V}_g = 2Re\Big\{\mathcal{M}^{\text{tree}\dagger}(A^+, B^-)\Big[\mathcal{M}(A^+, B^-) + \mathcal{M}(A^-, B^+) - \mathcal{M}(B^+, A^-) - \mathcal{M}(B^-, A^+)\Big]\Big\} \tag{187}$$

We will show later that considering one-loop amplitudes, the familiar virtual contribution is independent of the type of auxiliary partons, as this dependence drops out in combinations given by equations (186) and (187).

4.2 Unfamiliar virtual contribution

The phase space involved in the Born contribution is the same as for the virtual contribution. The difference in cross section is in the matrix element given by equation (160). We also observe that the on-shell limit $|k_T| \to 0$ taken before the Λ limit ($\Lambda \to \infty$) can be considered as the collinear limit of the auxiliary partons. Hence, it makes sense to look at the limit $k_T \to 0$ in order to pinpoint the unfamiliar contribution. We rewrite the momenta given by equation (18) to stick to the notation of auxiliary partons momenta given in [1]

$$k_1^\mu = -\Lambda P^\mu, \quad k_2^\mu = p_\Lambda^\mu = (\Lambda - x)P^\mu - k_T^\mu + \frac{|k_T|^2}{(\Lambda - x)\cdot 2P\cdot \overline{P}}\overline{P}^\mu. \tag{188}$$

Applying the on-shell limit to the dot product of the auxiliary partons, we obtain

$$2k_1\cdot k_2 = 2(-\Lambda P)\cdot p_\Lambda = \frac{-\Lambda}{\Lambda - x}|k_T|^2 \tag{189}$$

which straightforwardly goes to 0 in the limit $k_T \to 0$. We denote momentum k_1 as having negative energy, as this notation is common in the literature. There have been some works in the field of the collinear limit of one-loop amplitudes within color decomposition, as studied in [30, 31]. In these works, the collinear limit was studied on primitive amplitudes.

We will start with tree-level primitive amplitudes. We will see that the on-shell limit and Λ limit commute. With the formalism of [31], we write a collinear factorization formula, which at tree-level is

$$\mathcal{A}_{\text{tree}}(1^{h_1}, 2^{h_2}, 3, \cdots, n) \xrightarrow{1\|2} \sum_{h_i=\pm} \text{Split}^{\text{tree}}_{-h}(1^{h_1}, 2^{h_2})\mathcal{A}_{tree}(K^h, 3, \cdots, n) \tag{190}$$

where h denotes the helicities. The sum of the two initial state momenta k_1 and k_2 are typically represented by

$$k_1 \to zK, \quad k_2 \to (1-z)K. \tag{191}$$

Particularly in our study the spliting variable is not limited by 1, we have

$$K = -xP, \quad z = \frac{\Lambda}{x}. \tag{192}$$

As in [1], it will be clearer to introduce a very general notation with fractions x and y defined by

$$k_1 \to xK, \quad k_2 \to yK, \quad k_1 + k_2 = (x+y)K. \tag{193}$$

Within this notation we can study the collinear limit of any amplitude [32]. Nevertheless, let us consider the explicit example of the MHV amplitudes, with the momenta now denoted by p_i. For the auxiliary partons, we choose p_1 and p_2. The n-point gluon amplitudes for all configurations of the helicities of the 1st and 2nd gluons are

$$\mathcal{A}_n(1^-,2^+,3^-,4^+,\cdots,n^+) = \frac{\langle p_1p_3\rangle^4}{\langle p_1p_2\rangle\langle p_2p_3\rangle\langle p_3p_4\rangle\cdots\langle p_{n-1}p_n\rangle\langle p_np_1\rangle}. \tag{194}$$

$$\mathcal{A}_n(1^-,2^+,3^+,4^-,\cdots,n^-) = \frac{[p_3p_2]^4}{[p_1p_n][p_np_{n-1}]\cdots[p_4p_3][p_3p_2][p_2p_1]}. \tag{195}$$

$$\mathcal{A}_n(1^+,2^-,3^-,4^+,\cdots,n^+) = \frac{\langle p_2p_3\rangle^4}{\langle p_1p_2\rangle\langle p_2p_3\rangle\langle p_3p_4\rangle\cdots\langle p_{n-1}p_n\rangle\langle p_np_1\rangle}. \tag{196}$$

$$\mathcal{A}_n(1^+,2^-,3^+,4^-,\cdots,n^-) = \frac{[p_3p_1]^4}{[p_1p_n][p_np_{n-1}]\cdots[p_4p_3][p_3p_2][p_2p_1]}. \tag{197}$$

$$\mathcal{A}_n(1^+,2^+,3^-,4^-,\cdots,n^+) = \frac{\langle p_3p_4\rangle^4}{\langle p_1p_2\rangle\langle p_2p_3\rangle\langle p_3p_4\rangle\cdots\langle p_{n-1}p_n\rangle\langle p_np_1\rangle}. \tag{198}$$

$$\mathcal{A}_n(1^-,2^-,3^+,4^+,\cdots,n^-) = \frac{[p_4p_3]^4}{[p_1p_n][p_np_{n-1}]\cdots[p_4p_3][p_3p_2][p_2p_1]}. \tag{199}$$

We perform the substitution given in equation (193). The rules for scaling the spinors are given by

$$|xp\rangle = \sqrt{|x|}|p\rangle, \quad |xp] = \frac{|x|}{\sqrt{x}}|p] = \mathrm{sgn}(x)\sqrt{|x|}|p], \tag{200}$$

then applying all of this to the amplitudes (194) - (199) we obtain

$$\begin{aligned}\mathcal{A}_n\big(1^-,2^+,3^-,4^+,\cdots,n^+\big) \to &\frac{1}{\langle p_1p_2\rangle}\frac{x^2}{\sqrt{|xy|}}\frac{\langle pp_3\rangle^4}{\langle pp_3\rangle\langle p_3p_4\rangle\cdots\langle p_{n-1}p_n\rangle\langle p_np\rangle}\\ =&\frac{1}{\langle p_1p_2\rangle}\frac{x^2}{\sqrt{|xy|}|x+y|}\frac{\langle (x+y)pp_3\rangle^4}{\langle (x+y)pp_3\rangle\langle p_3p_4\rangle\cdots\langle p_{n-1}p_n\rangle\langle p_n(x+y)p\rangle}\\ =&\frac{1}{\langle p_1p_2\rangle}\frac{x^2}{\sqrt{|xy|}|x+y|}\mathcal{A}_{n-1}\big((x+y)p^-,3^-,4^+,\cdots,n^+\big),\end{aligned} \tag{201}$$

$$\begin{aligned}\mathcal{A}_n\big(1^-,2^+,3^+,4^-,\cdots,n^-\big) \to &\frac{1}{[p_2p_1]}\frac{\mathrm{sgn}(xy)y^2}{\sqrt{|xy|}}\frac{[p_3p]^4}{[pp_n][p_np_{n-1}]\cdots[p_4p_3][p_3p]}\\ =&\frac{-1}{[p_1p_2]}\frac{\mathrm{sgn}(xy)y^2}{\sqrt{|xy|}|x+y|}\frac{[(x+y)p_3p]^4}{[(x+y)pp_n][p_np_{n-1}]\cdots[p_4p_3][p_3(x+y)p]}\\ =&\frac{-1}{[p_1p_2]}\frac{\mathrm{sgn}(xy)y^2}{\sqrt{|xy|}|x+y|}\mathcal{A}_{n-1}\big((x+y)p^+,3^+,4^-,\cdots,n^-\big),\end{aligned} \tag{202}$$

$$
\begin{aligned}
\mathcal{A}_n\big(1^+,2^-,3^-,4^+,\cdots,n^+\big) \rightarrow & \frac{1}{\langle p_1p_2\rangle}\frac{y^2}{\sqrt{|xy|}}\frac{\langle pp_3\rangle^4}{\langle pp_3\rangle\langle p_3p_4\rangle\cdots\langle p_{n-1}p_n\rangle\langle p_np\rangle} \\
= & \frac{1}{\langle p_1p_2\rangle}\frac{y^2}{\sqrt{|xy|}|x+y|}\frac{\langle (x+y)pp_3\rangle^4}{\langle (x+y)pp_3\rangle\langle p_3p_4\rangle\cdots\langle p_{n-1}p_n\rangle\langle p_n(x+y)p\rangle} \\
= & \frac{1}{\langle p_1p_2\rangle}\frac{y^2}{\sqrt{|xy|}|x+y|}\mathcal{A}_{n-1}\big((x+y)p^-,3^-,4^+,\cdots,n^+\big),
\end{aligned}
\tag{203}
$$

$$
\begin{aligned}
\mathcal{A}_n\big(1^+,2^-,3^+,4^-,\cdots,n^-\big) \rightarrow & \frac{1}{[p_2p_1]}\frac{\mathrm{sgn}(xy)x^2}{\sqrt{|xy|}}\frac{[p_3p]^4}{[pp_n][p_np_{n-1}]\cdots[p_4p_3][p_3p]} \\
= & \frac{-1}{[p_1p_2]}\frac{\mathrm{sgn}(xy)x^2}{\sqrt{|xy|}|x+y|}\frac{[(x+y)p_3p]^4}{[(x+y)pp_n][p_np_{n-1}]\cdots[p_4p_3][p_3(x+y)p]} \\
= & \frac{-1}{[p_1p_2]}\frac{\mathrm{sgn}(xy)x^2}{\sqrt{|xy|}|x+y|}\mathcal{A}_{n-1}\big((x+y)p^+,3^+,4^-,\cdots,n^-\big),
\end{aligned}
\tag{204}
$$

$$
\begin{aligned}
\mathcal{A}_n\big(1^+,2^+,3^-,4^-,\cdots,n^+\big) \rightarrow & \frac{1}{\langle p_1p_2\rangle}\frac{1}{\sqrt{|xy|}}\frac{\langle p_3p_4\rangle^4}{\langle pp_3\rangle\langle p_3p_4\rangle\cdots\langle p_{n-1}p_n\rangle\langle p_np\rangle} \\
= & \frac{1}{\langle p_1p_2\rangle}\frac{|x+y|}{\sqrt{|xy|}}\frac{\langle p_3p_4\rangle^4}{\langle (x+y)pp_3\rangle\langle p_3p_4\rangle\cdots\langle p_{n-1}p_n\rangle\langle p_n(x+y)p\rangle} \\
= & \frac{1}{\langle p_1p_2\rangle}\frac{|x+y|}{\sqrt{|xy|}}\mathcal{A}_{n-1}\big((x+y)p^+,3^-,4^-,\cdots,n^+\big),
\end{aligned}
\tag{205}
$$

$$
\begin{aligned}
\mathcal{A}_n\big(1^-,2^-,3^+,4^+,\cdots,n^-\big) \rightarrow & \frac{1}{[p_2p_1]}\frac{\mathrm{sgn}(xy)}{\sqrt{|xy|}}\frac{[p_4p_3]^4}{[pp_n][p_np_{n-1}]\cdots[p_4p_3][p_3p]} \\
= & \frac{-1}{[p_1p_2]}\frac{\mathrm{sgn}(xy)|x+y|}{\sqrt{|xy|}}\frac{[p_4p_3]^4}{[(x+y)pp_n][p_np_{n-1}]\cdots[p_4p_3][p_3(x+y)p]} \\
= & \frac{-1}{[p_1p_2]}\frac{\mathrm{sgn}(xy)|x+y|}{\sqrt{|xy|}}\mathcal{A}_{n-1}\big((x+y)p^-,3^+,4^+,\cdots,n^-\big).
\end{aligned}
\tag{206}
$$

We observe that all results contain the $(n-1)$-point amplitude and a so-called splitting function. The splitting functions depend on the helicity of the auxiliary partons and the helicity of the off-shell gluon. We summarize

$$
\mathrm{Split}(g^-g^+\rightarrow g^-) = \frac{1}{\langle p_1p_2\rangle}\frac{x^2}{\sqrt{|xy|}|x+y|}, \tag{207}
$$

$$
\mathrm{Split}(g^-g^+\rightarrow g^+) = \frac{-1}{[p_1p_2]}\frac{\mathrm{sgn}(xy)y^2}{\sqrt{|xy|}|x+y|}, \tag{208}
$$

$$\mathrm{Split}(g^+g^- \to g^-) = \frac{1}{\langle p_1p_2\rangle}\frac{y^2}{\sqrt{|xy|}|x+y|}, \tag{209}$$

$$\mathrm{Split}(g^+g^- \to g^+) = \frac{-1}{[p_1p_2]}\frac{\mathrm{sgn}(xy)x^2}{\sqrt{|xy|}|x+y|}, \tag{210}$$

$$\mathrm{Split}(g^+g^+ \to g^+) = \frac{1}{\langle p_1p_2\rangle}\frac{|x+y|}{\sqrt{|xy|}}, \tag{211}$$

$$\mathrm{Split}(g^-g^- \to g^-) = \frac{-1}{[p_1p_2]}\frac{\mathrm{sgn}(xy)|x+y|}{\sqrt{|xy|}}. \tag{212}$$

At this point, we realized that the formulas (C.9) and (C.10) from [1] are incorrect, as they should have $|x+y|$ in the numerator. The correct ones are the formulas (211) and (212) given in this thesis.

The next step is to revert to the variables Λ and x within equations (191) and (192), which means exactly applying the auxiliary parton method's Λ limit. Namely, we make the following substitutions

$$x \to (\Lambda - x), \quad y \to -\Lambda, \tag{213}$$

which leads to

$$xy = -\Lambda \cdot (\Lambda - x), \quad x + y = -x. \tag{214}$$

Applying all of it to equations (207) - (212), and considering equation (167), we obtain for large Λ

$$\mathrm{Split}(g^-g^+ \to g^-) = \frac{1}{\kappa^*}\frac{\Lambda}{x}, \tag{215}$$

$$\mathrm{Split}(g^-g^+ \to g^+) = \frac{1}{\kappa}\frac{\Lambda}{x}, \tag{216}$$

$$\mathrm{Split}(g^+g^- \to g^-) = \frac{1}{\kappa^*}\frac{\Lambda}{x}, \tag{217}$$

$$\mathrm{Split}(g^+g^- \to g^+) = \frac{1}{\kappa}\frac{\Lambda}{x}, \tag{218}$$

and the last two vanish.

The on-shell limit of a tree-level amplitude was proven to be (see equation (5.4) from [16])

$$\lim_{|k_T|\to 0} \mathcal{A}_n^*(g^*, \chi) = \frac{|k_T|}{\kappa^*}\mathcal{A}_n(g^-, \chi) + \frac{|k_T|}{\kappa}\mathcal{A}_n(g^+, \chi). \tag{219}$$

Putting it together with the splittings from equations (215)-(218) and taking into account the Λ limit given by (21) or (161), with the understanding that there is a $|k_T|$ factor omitted, we obtain

$$\begin{aligned}\mathcal{A}_{\text{tree}}\big((-\Lambda P)^{h_1}, p_\Lambda^{h_2}, 3, \cdots, n\big) \xrightarrow{|k_T|\to 0} &\frac{\Lambda}{x\kappa}\mathcal{A}_{\text{tree}}\big((-xP)^{+}, 3, \cdots, n\big)\\ &+\frac{\Lambda}{x\kappa^*}\mathcal{A}_{\text{tree}}\big((-xP)^{-}, 3, \cdots, n\big) + \mathcal{O}\big(\Lambda^0\big),\end{aligned} \tag{220}$$

which indeed arises from the on-shell limit multiplied by the factor $\Lambda/(x_\chi|k_T|)$ from the Λ limit.

Following the formalism of [31], the factorization formula for one-loop amplitude is given by

$$\begin{aligned}\mathcal{A}_{\text{loop}}\big(1^{h_1}, 2^{h_2}, 3, \cdots, n\big) \xrightarrow{1\|2} \sum_{h_i=\pm} \Big\{&\text{Split}^{\text{tree}}_{-h}(1^{h_1}, 2^{h_2})\mathcal{A}_{\text{loop}}(K^h, 3, \cdots, n)\\ &+ \text{Split}^{\text{loop}}_{-h}(1^{h_1}, 2^{h_2})\mathcal{A}_{\text{tree}}(K^h, 3, \cdots, n)\Big\},\end{aligned} \tag{221}$$

with the one-loop splittings given by

$$\text{Split}^{\text{loop}}_{-h}(1^{h_1}, 2^{h_2}) = g_S^2 c_\Gamma \times \text{Split}^{\text{tree}}_{-h}(1^{h_1}, 2^{h_2}) \times \mathcal{V}(-h, 1^{h_1}, 2^{h_2}), \tag{222}$$

with $g_S^2 c_\Gamma$ given by equation (175).

We make a remark that the virtual contribution has to match the form of the real contribution, as mentioned in the introduction. Equation (221) shows the collinear limit of partons 1 and 2, which implies the limit $k_T \to 0$. We are looking for the non-smooth limit $k_T \to 0$, then it can not be the first term of equation (221) as the Split is at tree-level and $\mathcal{A}_{\text{loop}}$ is already $k_T = 0$. Thus, we turn our attention to the second term in equation (221). We will revisit this issue in the section where we introduce the real radiation contribution to the cross section.

We found that the functions $\mathcal{V}$ correspond to equations (B.3) and (B.6) in [31]. Specifically, for the process $g \to g_1, g_2$, we have

$$V_{g\to g_1,g_2}(h, 1^\pm, 2^\mp) = -\frac{1}{\epsilon^2}\left(\frac{\mu^2}{z(1-z)(-s_{12})}\right)^{\epsilon} + 2\ln z\,\ln(1-z) - \frac{\pi^2}{6}. \tag{223}$$

To maintain consistency with equation (4.12) in [1], we need to expand in ϵ and retain terms up to $\mathcal{O}(\epsilon^1)$. With some reorganization we can write

$$
\begin{aligned}
\mathcal{V}_{g\to g_1,g_2}(h,1^{\pm},2^{\mp}) &= \left(\frac{\mu^2}{-s_{12}}\right)^{\epsilon} \left\{ -\frac{1}{\epsilon^2}\left(\frac{1}{z(1-z)}\right)^{\epsilon} + 2\ln z\ \ln(1-z) - \frac{\pi^2}{6}\right\} + \mathcal{O}(\epsilon) \\
&= \left(\frac{\mu^2}{-s_{12}}\right)^{\epsilon} \left\{ -\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\ln z(1-z) - \frac{1}{2}\ln^2\frac{1}{z(1-z)} \right. \\
&\qquad \left. +2\ln\frac{1}{z}\ \ln\frac{1}{(1-z)} - \frac{\pi^2}{6}\right\} + \mathcal{O}(\epsilon) \\
&= \left(\frac{\mu^2}{-s_{12}}\right)^{\epsilon} \left\{ -\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\ln z(1-z) - \frac{1}{2}\ln^2\frac{1}{z} - \frac{1}{2}\ln^2\frac{1}{(1-z)} \right. \\
&\qquad \left. +\ln\frac{1}{z}\ \ln\frac{1}{(1-z)} - \frac{\pi^2}{6}\right\} + \mathcal{O}(\epsilon) \\
&= \left(\frac{\mu^2}{-s_{12}}\right)^{\epsilon} \left\{ -\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\ln z(1-z) - \frac{1}{2}\ln^2\frac{z}{(1-z)} - \frac{\pi^2}{6}\right\} + \mathcal{O}(\epsilon).
\end{aligned} \tag{224}
$$

In a similar manner, from equation (B.6) in [31], we can derive equation (4.11) in [1]. We write

$$
\begin{aligned}
\mathcal{V}_{g\to \bar{q}_1,q_2}(h,1^{\pm},2^{\mp}) = \left(\frac{\mu^2}{-s_{12}}\right)^{\epsilon} &\left\{ \frac{1}{\epsilon}\left(\ln z(1-z) + \frac{13}{6}\right) - \frac{1}{2}\ln^2\frac{z}{(1-z)} - \frac{\pi^2}{6} + \frac{83}{18} \right. \\
&\left. - \frac{\delta_R}{6} + \frac{1}{N_c^2}\left[\frac{1}{\epsilon^2} + \frac{3}{2}\frac{1}{\epsilon} + \frac{7+\delta_R}{2}\right] - \frac{n_f}{N_c}\left[\frac{2}{3}\frac{1}{\epsilon} + \frac{10}{9}\right]\right\} + \mathcal{O}(\epsilon),
\end{aligned} \tag{225}
$$

where the value of δ_R is equal 0 for dimensional reduction and 1 in the 't Hooft-Veltman scheme of dimensional regularization.

The collinear limit has to be taken after the invariant $s_{12} = (k_1 + k_2)^2$. We saw that at least at tree-level the large Λ limit and the collinear limit commute, and we also realized that the $\mathcal{V}$ functions do not depend on helicities, we can assume the unfamiliar part of one-loop amplitude to be

$$
\mathcal{A}^{*\mathrm{unf}}_{\mathrm{loop}}(-xP-k_T,3,\cdots,n) = \frac{a_\epsilon}{2}\mathcal{V}_{\mathrm{aux}}\mathcal{A}^{*}_{\mathrm{tree}}(-xP-k_T,3,\cdots,n). \tag{226}
$$

We just need to insert $\Lambda/x \to z$ into equations (224) and (225) to obtain (with $\delta_R = 1$)

$$
\mathcal{V}_{\text{aux-g}}(h,1^{\pm},2^{\mp}) = \left(\frac{\mu^2}{|k_T|^2}\right)^{\epsilon}\left\{\frac{2}{\epsilon}\ln\frac{\Lambda}{x} - i\pi - \frac{1}{\epsilon^2} + \frac{\pi^2}{3}\right\} + \mathcal{O}(\epsilon) + \mathcal{O}(\Lambda^{-1}), \tag{227}
$$

$$\mathcal{V}_{\text{aux-q}}(h, 1^{\pm}, 2^{\mp}) = \left(\frac{\mu^2}{|k_T|^2}\right)^{\epsilon} \left\{ \frac{2}{\epsilon}\ln\frac{\Lambda}{x} - i\pi + \frac{1}{\epsilon}\frac{13}{6} + \frac{\pi^2}{3} + \frac{80}{18} + \frac{1}{N_c^2}\left[\frac{1}{\epsilon^2} + \frac{3}{2}\frac{1}{\epsilon} + 4\right] \right.$$
$$\left. - \frac{n_f}{N_c}\left[\frac{2}{3}\frac{1}{\epsilon} + \frac{10}{9}\right] \right\} + \mathcal{O}(\epsilon) + \mathcal{O}(\Lambda^{-1}). \tag{228}$$

The $i\pi$ terms are present because of the negative argument in logarithms (as $\Lambda/x > 1$). To address this issue, we used a very helpful relation, specifically equation (29) in [14]. For any s, we write

$$\ln(s - i\eta) = \ln(-s - i\eta) + i\pi \mathrm{sgn}(s), \tag{229}$$

with a small positive η, its role is to avoid the branch of the logarithm. In the case of a positive argument of the logarithm, we can neglect it.

Equation (226) is at amplitude level, while we are looking for the matrix element formula. The color reference from the beginning of section 4.5 in [31], which points out that the leading partial amplitude retains the same color structure as the tree-level amplitude. Consequently, the virtual contribution to the cross section can be constructed with the same color structure as at tree-level. Therefore, we identify the unfamiliar virtual contribution as follows

$$dV^{\star\text{unf}} = a_\epsilon N_c \mathrm{Re}(\mathcal{V}_{\text{aux}}) dB^{\star}. \tag{230}$$

The factor of N_c arises from the leading partial amplitudes, while a factor of 2 comes from combining equation (160) with equation (226). We arrived at equation (226) from (221) by simply substituting the on-shell tree-level amplitudes with the off-shell ones. We would like to refer to section 4.1 of [1] for a justification for this substitution.

4.3 Familiar virtual contribution

In this section, we study the divergent structure of the familiar virtual contribution, which we define as

$$dV^{\star\text{fam}} = dV^{\star} - dV^{\star\text{unf}}. \tag{231}$$

The divergences present in one-loop amplitudes were studied in [33, 34]. Their universal structure is now well-known and commonly presented as a UV-subtracted contribution. Thus, we can write the entire divergent contribution, which is a sum of familiar and unfamiliar components, with the possibility to distinguish them. In this section, we will formulate virtual contributions as UV-subtracted, whereas in the previous section, the formulas were not UV-subtracted.

Particularly we will refer to the formula (13) of [34]. For auxiliary quarks, we consider only the terms significant for auxiliary partons. Keep in mind that the auxiliary parton method given by equation (21) is already applied. The space-like matrix element involves an off-shell gluon but carries the color representation from auxiliary quarks. For the divergent virtual contribution, we have the following formula with the omitted factor a_ϵ

$$
\begin{aligned}
&-2\frac{C_F}{\epsilon^2}|\overline{\mathcal{M}}^\star_{\text{aux-q}}|^2\\
&+\frac{2}{\epsilon}\left\{\ln\Big(\frac{\mu^2}{|k_T|^2}\Big)(\overline{\mathcal{M}}^\star_{\text{aux-q}})^2_{q\overline{q}}+\sum_{i\neq q,\overline{q}}\left[\ln\Big(\frac{\mu^2}{2\Lambda P\cdot p_i}\Big)(\overline{\mathcal{M}}^\star_{\text{aux-q}})^2_{iq}+\ln\Big(\frac{\mu^2}{-2\Lambda P\cdot p_i}\Big)(\overline{\mathcal{M}}^\star_{\text{aux-q}})^2_{i\overline{q}}\right]\right\}\\
&-2\frac{3C_F}{2\epsilon}|\overline{\mathcal{M}}^\star_{\text{aux-q}}|^2.
\end{aligned}
\tag{232}
$$

In that formula, we recognize two soft-collinear divergences in the first line and the rest of the soft terms in the second line, all associated with auxiliary quarks. In the third line, there are two collinear terms associated with auxiliary quarks. To move forward we need some relation that are obtained with the help of appendix B, particularly equation (B.14) of [1]

$$
|\overline{\mathcal{M}}^\star_{\text{aux-q}}|^2=|\overline{\mathcal{M}}^\star|^2,\quad(\overline{\mathcal{M}}^\star_{\text{aux-q}})^2_{q\overline{q}}=\frac{1}{2N_c}|\overline{\mathcal{M}}^\star|^2,\quad(\overline{\mathcal{M}}^\star_{\text{aux-q}})^2_{\overline{q}\overline{q}}=C_F|\overline{\mathcal{M}}^\star|^2.
\tag{233}
$$

We also will need the off-shell gluon correlator, which is independent of the type of auxiliary partons

$$
(\overline{\mathcal{M}}^\star)^2_{i\star}=(\overline{\mathcal{M}}^\star_{\text{aux-q}})^2_{iq}+(\overline{\mathcal{M}}^\star_{\text{aux-q}})^2_{i\overline{q}}.
\tag{234}
$$

With that, we can write the color conservation equation

$$
\sum_{i\neq q,\overline{q}}(\overline{\mathcal{M}}^\star_{\text{aux-q}})^2_{i\overline{q}}=-(\overline{\mathcal{M}}^\star_{\text{aux-q}})^2_{q\overline{q}}-(\overline{\mathcal{M}}^\star_{\text{aux-q}})^2_{\overline{q}\overline{q}}=-\left(\frac{1}{2N_c}+C_F\right)|\overline{\mathcal{M}}^\star|^2=-\frac{N_c}{2}|\overline{\mathcal{M}}^\star|^2,
\tag{235}
$$

The same result is obtained when q is substituted by $\overline{q}$. Thus, we rewrite equation (232) as follows

$$
\begin{aligned}
&-2\frac{C_F}{\epsilon^2}|\overline{\mathcal{M}}^\star|^2\\
&+\frac{1}{N_c\epsilon}\ln\Big(\frac{\mu^2}{|k_T|^2}\Big)|\overline{\mathcal{M}}^\star|^2+\frac{N_c}{\epsilon}\left[2\ln\Big(\frac{\Lambda}{x}\Big)-i\pi\right]|\overline{\mathcal{M}}^\star|^2+\frac{2}{\epsilon}\sum_{i\neq\star}\ln\Big(\frac{\mu^2}{2xP\cdot p_i}\Big)(\overline{\mathcal{M}}^\star)^2_{i\star}\\
&-3\frac{C_F}{\epsilon}|\overline{\mathcal{M}}^\star|^2,
\end{aligned}
\tag{236}
$$

where we introduced the momentum fraction x to ensure the presence of the xP component.

Due to color conservation, the last term in the second line acquires a minus sign, making the entire expression independent of x. We then reorganize to write

$$\begin{aligned}&-\frac{C_A}{\epsilon^2}|\overline{\mathcal{M}^\star}|^2+\frac{2}{\epsilon}\sum_{i\neq\star}\ln\Big(\frac{\mu^2}{2xP\cdot p_i}\Big)(\overline{\mathcal{M}^\star})^2_{i\star}-\frac{11N_c-2n_f}{6\epsilon}|\overline{\mathcal{M}^\star}|^2\\&+\frac{N_c}{\epsilon}\left\{\frac{1}{N_c^2}\left[\frac{1}{\epsilon}+\ln\Big(\frac{\mu^2}{|k_T|^2}\Big)\right]+2\ln\Big(\frac{\Lambda}{x}\Big)-i\pi+\frac{1}{3}+\frac{3}{2N_c^2}-\frac{n_f}{3N_c}\right\}|\overline{\mathcal{M}^\star}|^2.\end{aligned}\tag{237}$$

The divergent terms in the first line of this equation are referred to as the familiar virtual contribution. This contribution has exactly the same form as if the space-like gluon were in the on-shell limit, with the difference being the presence of an off-shell tree-level squared matrix element. The second line consists of the other virtual contribution, called the unfamiliar one. The terms with $1/\epsilon^2$ and $1/\epsilon$ are the same as in equation (228), after adding $\frac{11N_c-2n_f}{6\epsilon}$.

Considering auxiliary gluons instead of auxiliary quarks, we have a formula analogous to equation (232), with the gluons denoted by A and B

$$\begin{aligned}&-2\frac{C_A}{\epsilon^2}|\overline{\mathcal{M}^\star_{\text{aux-g}}}|^2\\&+\frac{2}{\epsilon}\left\{\ln\Big(\frac{\mu^2}{|k_T|^2}\Big)(\overline{\mathcal{M}^\star_{\text{aux-g}}})^2_{AB}+\sum_{i\neq A,B}\left[\ln\Big(\frac{\mu^2}{2\Lambda P\cdot p_i}\Big)(\overline{\mathcal{M}^\star_{\text{aux-g}}})^2_{iA}+\ln\Big(\frac{\mu^2}{-2\Lambda P\cdot p_i}\Big)(\overline{\mathcal{M}^\star_{\text{aux-g}}})^2_{iB}\right]\right\}\\&-2\frac{11N_c-2n_f}{6\epsilon}|\overline{\mathcal{M}^\star_{\text{aux-g}}}|^2.\end{aligned}\tag{238}$$

The formulas analogues to equations (233), (234) and (235) for auxiliary gluons are

$$|\overline{\mathcal{M}^\star_{\text{aux-g}}}|^2=|\overline{\mathcal{M}^\star}|^2,\quad(\overline{\mathcal{M}^\star_{\text{aux-g}}})^2_{AB}=-\frac{N_c}{2}|\overline{\mathcal{M}^\star}|^2,\quad(\overline{\mathcal{M}^\star_{\text{aux-g}}})^2_{BB}=C_A|\overline{\mathcal{M}^\star}|^2,\tag{239}$$

$$(\overline{\mathcal{M}^\star})^2_{i\star}=(\overline{\mathcal{M}^\star_{\text{aux-g}}})^2_{iA}+(\overline{\mathcal{M}^\star_{\text{aux-g}}})^2_{iB},\tag{240}$$

$$\sum_{i\neq AB}(\overline{\mathcal{M}^\star_{\text{aux-g}}})^2_{iB}=-\Big(-\frac{N_c}{2}+C_A\Big)|\overline{\mathcal{M}^\star}|^2=-\frac{N_c}{2}|\overline{\mathcal{M}^\star}|^2.\tag{241}$$

Then equation (238) become

$$
\begin{aligned}
&- 2\frac{C_A}{\epsilon^2}|\overline{\mathcal{M}}^{\star}|^2 \\
&- \frac{N_c}{\epsilon}\ln\Big(\frac{\mu^2}{|k_T|^2}\Big)|\overline{\mathcal{M}}^{\star}|^2 + \frac{N_c}{\epsilon}\Big[\ln\Big(\frac{\Lambda}{x}\Big) - i\pi\Big]|\overline{\mathcal{M}}^{\star}|^2 + \frac{2}{\epsilon}\sum_{i\neq\star}\ln\Big(\frac{\mu^2}{2xP\cdot p_i}\Big)(\overline{\mathcal{M}}^{\star})^2_{i\star} \\
&- \frac{11N_c - 2n_f}{3\epsilon}|\overline{\mathcal{M}}^{\star}|^2,
\end{aligned}
\tag{242}
$$

and after reorganization

$$
\begin{aligned}
&- \frac{C_A}{\epsilon^2}|\overline{\mathcal{M}}^{\star}|^2 + \frac{2}{\epsilon}\sum_{i\neq\star}\ln\Big(\frac{\mu^2}{2xP\cdot p_i}\Big)(\overline{\mathcal{M}}^{\star})^2_{i\star} - \frac{11N_c - 2n_f}{6\epsilon}|\overline{\mathcal{M}}^{\star}|^2 \\
&- \frac{N_c}{\epsilon}\Big\{\frac{1}{\epsilon} + \ln\Big(\frac{\mu^2}{|k_T|^2}\Big) - 2\ln\Big(\frac{\Lambda}{x}\Big) + i\pi\Big\}|\overline{\mathcal{M}}^{\star}|^2 - \frac{11N_c - 2n_f}{6\epsilon}|\overline{\mathcal{M}}^{\star}|^2.
\end{aligned}
\tag{243}
$$

We recognize the same structure as in equation (237). The familiar virtual contribution for auxiliary gluons is exactly the same as for auxiliary quarks, while the unfamiliar virtual contribution depends on the type of auxiliary partons. Nevertheless, the terms with $1/\epsilon^2$ and $1/\epsilon$ are identical to those form equation (227), after adding $\frac{11N_c-2n_f}{6\epsilon}$.

Note that the unfamiliar contribution differs from the formulas addressed in section 4.2 by the UV divergent terms, as mentioned at the beginning of this section. We will see later that both approaches are consistent.

5 Unfamiliar real and virtual contribution

In the previous section, we examined the structure of the virtual contribution, which consisted of familiar and unfamiliar parts. The divergences in the familiar virtual contributions align with the structure known from collinear factorization, and established methods exist to handle these. However, the unfamiliar virtual contribution presents two main issues that need to be addressed. The first issue is the presence of $\ln(\Lambda)$, which leads to a divergence as $\Lambda \to \infty$. The second issue is that the unfamiliar contribution depends on the type of auxiliary partons used in the method to obtain an off-shell amplitude. In the following, we will establish the formulas for the real contribution to address the mentioned issues. We will find that the real contribution can also be divided into two parts: the unfamiliar and the familiar real contribution.

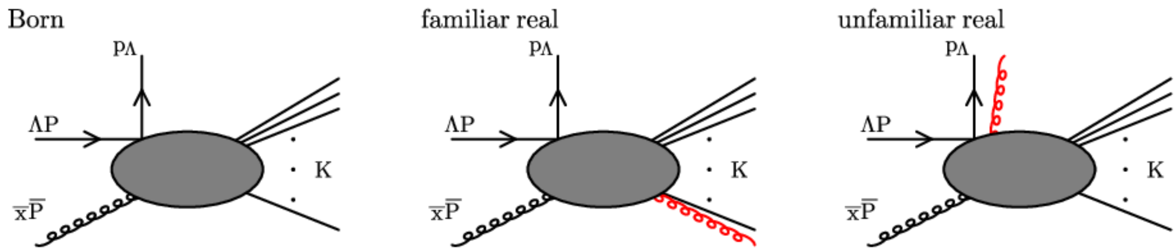


Figure 7: The kinematic difference between familiar and unfamiliar real contributions. The red gluon is the radiated gluon.

While the familiar real contribution was defined in equation (36), to introduce the unfamiliar real contribution, we will start with the simple process involving auxiliary quarks and no final state gluons at the Born level. The origin of the unfamiliar real contribution comes directly from the region in the phase space where the radiated gluon takes part in the consumption of the large parameter Λ, as shown in figure 7.

Region A, from the figure 6, corresponds to the situation where the radiative gluon does not take part in the consumption of the parameter Λ, and we would like to include this region in the familiar real contribution. In region B, there is no difference between familiar and unfamiliar contributions from figure 7, regardless of the total number of final state partons. In other words, the unfamiliar real contribution when the radiative gluon is collinear to the off-shell gluon looks like the Born level. Thus, we need to keep in mind that this region can cause double counting and we will address this issue later. This restriction was also mentioned in [35] from the point of view of angular ordering. The definition of the unfamiliar real contribution, which will be given in the next subsection, is restricted to regions C_1 and C_2.

5.1 Completed real contribution

The decomposition of the radiative gluon is given by equation (25). The definition of the unfamiliar real contribution requires the modification of the auxiliary parton method, called the triple Λ limit. The more detailed derivation of this limit is left to appendix B. Here, we show the parametrization of the auxiliary parton momenta and the radiation momentum. The parametrization is as follows

$$k_{in}^{\mu} = (x+\Lambda)P^{\mu}, \quad q^{\mu} = -x_q\Lambda P^{\mu} + q_T^{\mu} - \frac{q_T^2}{\nu^2 x_q \Lambda}\overline{P}^{\mu}, \quad r^{\mu} = -x_r\Lambda P^{\mu} + r_T^{\mu} - \frac{r_T^2}{\nu^2 x_r \Lambda}\overline{P}^{\mu}, \tag{244}$$

with

$$x_r + x_q = 1. \tag{245}$$

We see that for unfamiliar real contribution the x_r form figure 6 is limited by b = 1. The auxiliary finial state momentum is denoted by q. Then up to $\mathcal{O}(\Lambda^{-1})$ we have

$$k_{in}^{\mu} + q^{\mu} + r^{\mu} = xP^{\mu} + q_T^{\mu} + r_T^{\mu} = xP^{\mu} + k_T^{\mu}. \tag{246}$$

With that, we define the unfamiliar real contribution as

$$\begin{aligned} &d\mathrm{R}^{\star\mathrm{unf}}\big(k_{\mathrm{in}}, k_{\overline{\mathrm{in}}}; \{p_i\}_n\big) \overset{\Lambda\to\infty}{=} \\ &\frac{a_\epsilon \mu^{2\epsilon}}{\pi_\epsilon}\frac{2x}{C_{\mathrm{aux}}}\int d^{4-2\epsilon}q\,\delta(q^2)d^{4-2\epsilon}r\,\delta(r^2)\delta(\Lambda - \Lambda x_q - \Lambda x_r)\delta^{2-2\epsilon}(k_T + q_T + r_T) \\ &\qquad\qquad \times \frac{d\Sigma_{n+2}^{\mathrm{aux}}}{dqdr}\big((\Lambda+x)P, k_{\overline{\mathrm{in}}}; r, q, \{p_i\}_n\big) J_{\mathrm{B}}\big(\{p_i\}_n\big). \end{aligned} \tag{247}$$

Comparing to equation (21), we integrate over the auxiliary momenta q and r to encompass the proper phase space (remember that for the radiative gluon we have restrictions mentioned earlier) and define a cross section. The delta function with the argument $\Lambda - \Lambda x_q - \Lambda x_r$ expresses the possibility for both final auxiliary partons to have arbitrarily large P components. The first $1/\Lambda$ factor originates from the flux factor of $d\Sigma_{n+2}^{\mathrm{aux}}$. In the original auxiliary parton method with two auxiliary partons, the second $1/\Lambda$ factor would come from the $1/2x_q$ in the differential volume of q, as you can see in equations (33) and (32). However, since we have three momenta parametrized with Λ, we rather extract $1/\Lambda$ from the $\delta(\Lambda - \Lambda x_q - \Lambda x_r)$ function. The factor of 2 corrects the factor of $1/2$ from equation (32). Note also that dimensional regularization is already applied here.

For the pure gluonic process, the final auxiliary gluon can correspond to any of the final state gluons. Thus, we assume that the symmetry factor $\frac{1}{n_g!}$ is included in Σ_{n+1}^{aux} from the outset.

We will proceed by applying the Sudakov decomposition to the momenta q and r, and rewrite equation (247) using the relation (32) for dq and dr. We write

$$
\begin{aligned}
&d\mathrm{R}^{\star\mathrm{unf}}\big(k_{\mathrm{in}},k_{\overline{\mathrm{in}}};\{p_i\}_n\big) \overset{\Lambda\to\infty}{=} \\
&\qquad\frac{a_\epsilon\mu^{2\epsilon}}{\pi_\epsilon}\frac{2x}{C_{\mathrm{aux}}}\int_0^1\frac{dx_q}{2x_q}\int_0^{\overline{x}}d\overline{x}_q\int d^{2-2\epsilon}q_T\,\delta\Big(\overline{x}_q-\frac{|q_T|^2}{\nu^2x_q\Lambda}\Big) \\
&\quad\times\left[\int_0^{x_C}\frac{dx_r}{2x_r}\int_{D(r_T)x_r}^{Cx_r}d\overline{x}_r+\int_{x_C}^1\frac{dx_r}{2x_r}\int_{D(r_T)x_r}^{\overline{x}}d\overline{x}_r\right]\int d^{2-2\epsilon}r_T\,\delta\Big(\overline{x}_r-\frac{|r_T|^2}{\nu^2x_r\Lambda}\Big) \\
&\qquad\times\Lambda^{-1}\delta(1-x_q-x_r)\delta^{2-2\epsilon}(k_T+q_T+r_T) \\
&\times\frac{d\Sigma^{\mathrm{aux}}_{n+2}}{dqdr}\Big(\big(x+\Lambda\big)P,\overline{x};-x_r\Lambda P+r_T+\overline{x}_r\overline{P},-x_q\Lambda P+q_T+\overline{x}_q\overline{P},\{p_i\}_n\Big)J_{\mathrm{B}}\big(\{p_i\}_n\big).
\end{aligned}
\tag{248}
$$

The sum of the integrals in the square bracket correspond to the sum of the C regions of figure 6. With that, we ensure the hard upper limit $\overline{x}_r < \overline{x}$, which was missed in [1]. Fortunately, this inaccuracy coincidentally did not affect the final result. The C constant came from equation (26), while D from equation (27). We would like to mention here that the second constant will be dependent on r_T by $D = D(r_T)$. We observe that there are three partons carrying a large P component. Also, note that $\overline{x}_q$ and $\overline{x}_r$ are suppressed by Λ to ensure the on-shellness of the auxiliary partons. The triple Λ limit was identified in appendix B to be

$$
\begin{aligned}
&\frac{1}{C_{\mathrm{aux}}}|\overline{\mathcal{M}}^{\mathrm{aux}}|^2\big((x+\Lambda)P,k_{\overline{\mathrm{in}}};-x_r\Lambda P+r_T+\overline{x}_r\overline{P},-x_q\Lambda P+q_T+\overline{x}_q\overline{P},\{p_i\}_n\big) \\
&\qquad\xrightarrow{\Lambda\to\infty}\mathcal{Q}_{\mathrm{aux}}(x_q,q_T,x_r,q_T)\frac{\Lambda^2|\overline{\mathcal{M}}^{\star}|^2\big(xP+q_T+r_T,k_{\overline{\mathrm{in}}};\{p_i\}_n\big)}{x^2|q_T+r_T|^2},
\end{aligned}
\tag{249}
$$

with

$$
\begin{aligned}
&\mathcal{Q}_{\mathrm{aux}}(x_q,q_T,x_r,q_T) = x_qx_r\mathcal{P}_{\mathrm{aux}}(x_q,x_r)|q_T+r_T|^2 \\
&\times\left[\frac{c_{\overline{q}}}{|q_T|^2|r_T|^2}+\frac{1}{x_r|q_T|^2+x_q|r_T|^2-x_qx_r|q_T+r_T|^2}\left(\frac{c_rx_r^2}{|r_T|^2}+\frac{c_qx_q^2}{|q_T|^2}\right)\right].
\end{aligned}
\tag{250}
$$

The quantities dependent of auxiliary partons are

$$
\text{auxiliary quarks:}\quad \mathcal{P}_{\mathrm{aux}}(x_q,x_r)=\frac{1+x_q^2-\epsilon x_r^2}{x_r},\quad c_q=c_{\overline{q}}=N_c,\quad c_r=\frac{-1}{N_c},\tag{251}
$$

$$
\text{auxiliary gluons:}\quad \mathcal{P}_{\mathrm{aux}}(x_q,x_r)=\frac{1+x_q^4+x_r^4}{x_qx_r},\quad c_q=c_{\overline{q}}=c_r=N_c,\tag{252}
$$

$$
\text{gluon to quarks:}\quad \mathcal{P}_{\mathrm{aux}}(x_q,x_r)=1-\frac{2x_qx_r}{1-\epsilon},\quad c_q=c_r=\frac{1}{2},\quad c_{\overline{q}}=\frac{-1}{2N_c^2}.\tag{253}
$$

The last equation refers to the process with one auxiliary gluon in the initial state and a quark pair in the final state. Labels q and r refer to the final state partons.

Applying equation (249) to formula (248) and not forgetting about the flux factor from equation (33), we obtain

$$
\begin{aligned}
&d\mathrm{R}^{\star\mathrm{unf}}\big(k_{\mathrm{in}}, k_{\overline{\mathrm{in}}}; \{p_i\}_n\big) \xrightarrow{\Lambda\to\infty} \\
&\qquad \frac{2a_\epsilon}{\pi_\epsilon \mu^{\bar\epsilon}} \int_0^1 \frac{dx_q}{2x_q} \int_0^{\bar x} d\bar x_q \int d^{2+\bar\epsilon} q_T \delta\bigg(\bar x_q - \frac{|q_T|^2}{\nu^2 x_q \Lambda}\bigg) \\
&\times \bigg[\int_0^{x_C} \frac{dx_r}{2x_r} \int_{D(r_T)x_r}^{Cx_r} d\bar x_r + \int_{x_C}^1 \frac{dx_r}{2x_r} \int_{D(r_T)x_r}^{\bar x} d\bar x_r\bigg] \int d^{2-2\epsilon} r_T \delta\bigg(\bar x_r - \frac{|r_T|^2}{\nu^2 x_r \Lambda}\bigg) \\
&\qquad \times \Lambda^{-1}\delta(1-x_q-x_r)\delta^{2+\bar\epsilon}(k_T+q_T+r_T) \\
&\qquad \times \mathcal{Q}_{\mathrm{aux}}(x_q, q_T, x_r, q_T) \frac{x^2}{\Lambda} \frac{\Lambda^2 d\Sigma_n^\star\big(xP+k_T, \bar x; \{p_i\}_n\big)}{x^2|k_T|^2} J_{\mathrm{B}}\big(\{p_i\}_n\big).
\end{aligned}
\tag{254}
$$

After calculating the $\bar x_q$ integral, we find

$$
d\mathrm{R}^{\star\mathrm{unf}}\big(k_{\mathrm{in}}, k_{\overline{\mathrm{in}}}; \{p_i\}_n\big) = \mathcal{R}_{\mathrm{aux}} d\mathrm{B}^\star\big(k_{\mathrm{in}}, k_{\overline{\mathrm{in}}}; \{p_i\}_n\big), \tag{255}
$$

with

$$
\begin{aligned}
\mathcal{R}_{\mathrm{aux}}(x, k_T, \mu, \Lambda, \bar x) &= \frac{2a_\epsilon}{\pi_\epsilon \mu^{\bar\epsilon}} \int_0^1 \frac{dx_q}{2x_q} \int d^{2+\bar\epsilon} q_T \theta(0 < |q_T|^2 < \nu^2 \bar x x_q \Lambda) \\
\times \bigg[\int_0^{x_C} \frac{dx_r}{2x_r} \int_{D(r_T)x_r}^{Cx_r} d\bar x_r &+ \int_{x_C}^1 \frac{dx_r}{2x_r} \int_{D(r_T)x_r}^{\bar x} d\bar x_r\bigg] \int d^{2-2\epsilon} r_T \delta\bigg(\bar x_r - \frac{|r_T|^2}{\nu^2 x_r \Lambda}\bigg) \\
&\times \delta(1-x_q-x_r)\delta^{2+\bar\epsilon}(k_T+q_T+r_T)\mathcal{Q}_{\mathrm{aux}}(x_q, q_T, x_r, q_T).
\end{aligned}
\tag{256}
$$

Momentum conservation implies $k_T + q_T + r_T = 0$ and also the fact that the sum of x_q and x_r is equal to 1, allows us to write

$$
\begin{aligned}
\mathcal{Q}_{\mathrm{aux}}(x_q, q_T, x_r, q_T) = x_q x_r \mathcal{P}_{\mathrm{aux}}(x_q, x_r)|k_T|^2 \bigg(& \frac{c_{\bar q}}{|r_T|^2 |r_T + k_T|^2} \\
&+ \frac{c_q (1-x_r)^2}{|r_T + k_T|^2 |r_T + x_r k_T|^2} + \frac{c_r x_r^2}{|r_T|^2 |r_T + x_r k_T|^2}\bigg),
\end{aligned}
\tag{257}
$$

which straightforwardly leads to the cancellation of $1/x_q$ and $1/x_r$ in the integral. Then, the only potential singularities caused by x_q and x_r could be in $\mathcal{P}_{\mathrm{aux}}(x_q, x_r)$. When we use quarks as auxiliary partons, the only singularity is due to x_r. For auxiliary gluons, we demand x_r to be smaller than x_q, which ensures that the potential singularity with respect to x_q is no longer valid. As we distinguish the auxiliary gluons, we have to remember not to include the factor of 2. With this knowledge and the fact that q_T and r_T integrals are UV finite, we can take the Λ limit inside the function $\theta(0 < |q_T|^2 < \nu^2 \bar x x_q \Lambda)$, which leads us to the elimination of the q_T

and x_q integrals. Then we obtained

$$
\begin{aligned}
\mathcal{R}_{\text{aux}}(x, k_T, \mu, \Lambda, \overline{x}) = & \frac{a_\epsilon}{2\pi_\epsilon \mu^{\overline{\epsilon}}} \\
& \times \left[\int_0^{x_C} dx_r \mathcal{P}_{\text{aux}}(x_r) \int_{D(r_T)x_r}^{Cx_r} d\overline{x}_r + \int_{x_C}^{u} dx_r \mathcal{P}_{\text{aux}}(x_r) \int_{D(r_T)x_r}^{\overline{x}} d\overline{x}_r \right] \\
& \times \int d^{2-2\epsilon} r_T \delta\left(\overline{x}_r - \frac{|r_T|^2}{\nu^2 x_r \Lambda} \right) \\
& \times |k_T|^2 \left(\frac{c_{\overline{q}}}{|r_T|^2 |r_T + k_T|^2} + \frac{c_q (1-x_r)^2}{|r_T + k_T|^2 |r_T + x_r k_T|^2} + \frac{c_r x_r^2}{|r_T|^2 |r_T + x_r k_T|^2} \right).
\end{aligned}
\tag{258}
$$

The upper integration limit u takes the value of 1 for auxiliary quarks and 1/2 for auxiliary gluons. At this point, we will introduce different notation compared to the one used in [1] to organize the singular structure of equation (258) in a more convenient way. We then write

$$
\begin{aligned}
\mathcal{R}_{\text{aux}}(x, k_T, \mu, \Lambda, \overline{x}) = \frac{a_\epsilon}{2\pi_\epsilon \mu^{\overline{\epsilon}}} & \left[\int_0^1 dx_r \mathcal{P}_{\text{aux}}(x_r) \int_{D(r_T)x_r}^{Cx_r} d\overline{x}_r \right. \\
& + \int_{x_C}^{u} dx_r \mathcal{P}_{\text{aux}}(x_r) \int_{D(r_T)x_r}^{\overline{x}} d\overline{x}_r \\
& \left. - \int_{x_C}^{1} dx_r \mathcal{P}_{\text{aux}}(x_r) \int_{D(r_T)x_r}^{Cx_r} d\overline{x}_r \right] \\
\times \int d^{2-2\epsilon} r_T \delta\left(\overline{x}_r - \frac{|r_T|^2}{\nu^2 x_r \Lambda} \right) & \\
\times |k_T|^2 \Big[(c_{\overline{q}} + c_q) T_{\overline{q}} + c_q (T_q - T_{\overline{q}}) + c_r T_r \Big], &
\end{aligned}
\tag{259}
$$

with

$$
T_{\overline{q}} = \frac{1}{|r_T|^2 |r_T + k_T|^2}, \tag{260}
$$

$$
T_q = \frac{(1-x_r)^2}{|r_T + k_T|^2 |r_T + x_r k_T|^2}, \tag{261}
$$

$$
T_r = \frac{x_r^2}{|r_T|^2 |r_T + x_r k_T|^2}. \tag{262}
$$

It is straightforward to realize that only the first term in the last line of equation (259) causes a singularity of $\mathcal{R}_{\text{aux}}$ in the limit $x_r \to 0$, while the other two do not. We also can extract the pole explicitly from $\mathcal{P}_{\text{aux}}(x_q, x_r)$ by

$$
\mathcal{P}_{\text{aux}}(x_q, x_r) = \frac{\mathcal{P}_{\text{aux}}^{\text{pole}}}{x_r} + \mathcal{P}_{\text{aux}}^{\text{rest}}(x_r) \tag{263}
$$

We extract the dependence on u

$$
\begin{aligned}
\mathcal{R}_{\text{aux}}(x, k_T, \mu, \Lambda, \overline{x}) =& \frac{a_\epsilon}{2\pi_\epsilon \mu^{\overline{\epsilon}}} \int_0^1 dx_r \frac{\mathcal{P}_{\text{aux}}^{\text{pole}}}{x_r} \int_{D(r_T)x_r}^{Cx_r} d\overline{x}_r \int d^{2+\overline{\epsilon}} r_T \delta\left(\overline{x}_r - \frac{|r_T|^2}{\nu^2 x_r \Lambda}\right) \\
& \times |k_T|^2 (c_{\overline{q}} + c_q) T_{\overline{q}} \\
& + \frac{a_\epsilon}{2\pi_\epsilon \mu^{\overline{\epsilon}}} \left[\int_{x_C}^{u} dx_r \frac{\mathcal{P}_{\text{aux}}^{\text{pole}}}{x_r} \int_{D(r_T)x_r}^{\overline{x}} d\overline{x}_r \right. \\
& \left. - \int_{x_C}^{1} dx_r \frac{\mathcal{P}_{\text{aux}}^{\text{pole}}}{x_r} \int_{D(r_T)x_r}^{Cx_r} d\overline{x}_r \right] \\
& \times \int d^{2+\overline{\epsilon}} r_T \delta\left(\overline{x}_r - \frac{|r_T|^2}{\nu^2 x_r \Lambda}\right) |k_T|^2 (c_{\overline{q}} + c_q) T_{\overline{q}} \\
& + \frac{a_\epsilon}{2\pi_\epsilon \mu^{\overline{\epsilon}}} \left[\int_0^{x_C} dx_r \frac{\mathcal{P}_{\text{aux}}^{\text{pole}}}{x_r} \int_{D(r_T)x_r}^{Cx_r} d\overline{x}_r \right. \\
& \left. + \int_{x_C}^{u} dx_r \frac{\mathcal{P}_{\text{aux}}^{\text{pole}}}{x_r} \int_{D(r_T)x_r}^{\overline{x}} d\overline{x}_r \right] \\
& \times \int d^{2+\overline{\epsilon}} r_T \delta\left(\overline{x}_r - \frac{|r_T|^2}{\nu^2 x_r \Lambda}\right) |k_T|^2 \left[c_q (T_q - T_{\overline{q}}) + c_r T_r \right] \\
& + \frac{a_\epsilon}{2\pi_\epsilon \mu^{\overline{\epsilon}}} \left[\int_0^{x_C} dx_r \mathcal{P}_{\text{aux}}^{\text{rest}}(x_r) \int_{D(r_T)x_r}^{Cx_r} d\overline{x}_r \right. \\
& \left. + \int_{x_C}^{u} dx_r \mathcal{P}_{\text{aux}}^{\text{rest}}(x_r) \int_{D(r_T)x_r}^{\overline{x}} d\overline{x}_r \right] \\
& \times \int d^{2+\overline{\epsilon}} r_T \delta\left(\overline{x}_r - \frac{|r_T|^2}{\nu^2 x_r \Lambda}\right) \\
& \times |k_T|^2 \left[(c_{\overline{q}} + c_q) T_{\overline{q}} + c_q (T_q - T_{\overline{q}}) + c_r T_r \right],
\end{aligned}
\tag{264}
$$

and we see that only the first term does not avoid the pole.

Hence, we perform the following decomposition to separate the parts with and without the pole in x_r in equation (264)

$$
\mathcal{R}_{\text{aux}} = \mathcal{R}_{\text{aux}}^{\text{pole}} + \mathcal{R}_{\text{aux}}^{\text{rest}}. \tag{265}
$$

While performing the calculation of $\mathcal{R}_{\text{aux}}^{\text{rest}}$, the limit $\Lambda \to \infty$ can be taken straightforwardly. For detailed derivations of equation (263) and $\mathcal{R}_{\text{aux}}^{\text{rest}}$, we refer the reader to appendix C. Here, we will proceed with $\mathcal{R}_{\text{aux}}^{\text{pole}}$. We begin by choosing the phase space cut to match the condition the same as given in [35], which we will obtain in equation (267). Therefore, we define the constant $D = D(r_T)$ to be

$$
D(r_T) = \frac{|r_T + k_T|^2}{\nu^2 \Lambda}. \tag{266}
$$

With this we can perform the $\overline{x}_r$ integral to get

$$
\begin{aligned}
\mathcal{R}_{\text{aux}}^{\text{pole}} =& \frac{a_\epsilon}{2\pi_\epsilon \mu^{\overline{\epsilon}}}|k_T|^2(c_{\overline{q}}+c_q) \\
&\times \int_0^1 dx_r \frac{\mathcal{P}_{\text{aux}}^{\text{pole}}}{x_r} \int d^{2+\overline{\epsilon}} r_T \frac{\theta(x_r^2|r_T+k_T|^2 < |r_T|^2 < \nu^2 C x_r^2 \Lambda)}{|r_T|^2|r_T+k_T|^2}, \\
=& \frac{a_\epsilon}{2\pi_\epsilon \mu^{\overline{\epsilon}}}|k_T|^2(c_{\overline{q}}+c_q) \\
&\times \int_0^1 dx_r \frac{\mathcal{P}_{\text{aux}}^{\text{pole}}}{x_r} \int d^{2+\overline{\epsilon}} r_T \frac{1}{|r_T|^2|r_T+k_T|^2} \theta\left(\frac{|r_T|}{\nu\sqrt{C\Lambda}} < x_r < \frac{|r_T|}{|r_T+k_T|}\right).
\end{aligned}
\tag{267}
$$

With this choice of D we can solve the integrals using equation (367), resulting in the following expression

$$
\mathcal{R}_{\text{aux}}^{\text{pole}} = a_\epsilon \left(\frac{|k_T|}{\mu}\right)^{\overline{\epsilon}} (c_{\overline{q}}+c_q)\mathcal{P}_{\text{aux}}^{\text{pole}} \frac{1}{\overline{\epsilon}} \ln \frac{\nu^2 C\Lambda}{|k_T|^2} + \mathcal{O}(\overline{\epsilon}) + \mathcal{O}(\Lambda^{-1}). \tag{268}
$$

We realize that for the pole contribution, we have $c_{\overline{q}} = c_q = N_c$ and $\mathcal{P}_{\text{aux}}^{\text{pole}} = 2$, which is independent of the type of auxiliary partons.

$$
\mathcal{R}_{\text{aux}}^{\text{pole}} = \frac{4a_\epsilon N_c}{\overline{\epsilon}} \left(\frac{|k_T|}{\mu}\right)^{\overline{\epsilon}} \ln \frac{\nu^2 C\Lambda}{|k_T|^2} + \mathcal{O}(\overline{\epsilon}) \tag{269}
$$

Collecting all together we find a structure

$$
\mathcal{R}_{\text{aux}} = \mathcal{R}_{\text{univ}}^{\text{pole}} + \mathcal{R}_{\text{aux}}^{\text{rest}}, \tag{270}
$$

where components are defined in appendix C and in equation (269). For auxiliary quarks with equation (351) we obtain

$$
\mathcal{R}_{\text{aux-q}} = \mathcal{R}_{\text{aux}}^{\text{pole}} + 2a_\epsilon N_c \left(\frac{|k_T|}{\mu}\right)^{\overline{\epsilon}} \left\{ -\frac{3}{\overline{\epsilon}} - \frac{\pi^2}{3} + \frac{5+2\delta}{4} - \frac{1}{N_c^2}\left[\frac{2}{\overline{\epsilon}^2} - \frac{3}{2\overline{\epsilon}} + \frac{7+\delta}{4}\right]\right\}. \tag{271}
$$

For auxiliary gluons we include equation (357) to get

$$
\mathcal{R}_{\text{aux-g}} = \mathcal{R}_{\text{aux}}^{\text{pole}} + 2a_\epsilon N_c \left(\frac{|k_T|}{\mu}\right)^{\overline{\epsilon}} \left\{ \frac{2}{\overline{\epsilon}^2} - \frac{1}{\overline{\epsilon}}\frac{11}{2} - \frac{\pi^2}{3} + \frac{67}{18}\right\}. \tag{272}
$$

And finally for $\mathcal{R}_{\text{aux-gq}}$ there is only last therm present given by equation (360), then we write

$$
\mathcal{R}_{\text{aux-gq}} = 2a_\epsilon \left(\frac{|k_T|}{\mu}\right)^{\overline{\epsilon}} \left\{ \frac{2}{3\overline{\epsilon}} - \frac{13}{18} + \frac{\delta}{6} - \frac{1}{N_c^2}\left(\frac{1}{3\overline{\epsilon}} + \frac{\delta}{12}\right)\right\}. \tag{273}
$$

5.2 Completion of the NLO contribution

In the previous sections 4.2 and 5.1, we obtained both the unfamiliar real and unfamiliar virtual contributions to the cross section at NLO. We realized that each unfamiliar contribution violates auxiliary parton universality, the smooth on-shell limit, and the smooth large Λ limit. In the following section, we will complete the unfamiliar contribution and attempt to provide some interpretations.

The unfamiliar real contribution is calculated in section 5.1, and completed result is given by

$$d\mathrm{R}^{\star\mathrm{unf}}\big(k_{\mathrm{in}}, k_{\overline{\mathrm{in}}}; \{p_i\}_n\big) = \mathcal{R}_{\mathrm{aux}} d\mathrm{B}^{\star}\big(k_{\mathrm{in}}, k_{\overline{\mathrm{in}}}; \{p_i\}_n\big). \tag{274}$$

For auxiliary quarks $\mathcal{R}_{\mathrm{aux}}$ is formulated by equation (271), which with $\bar{\epsilon} = -2\epsilon$ and $\delta = 1$ is given by

$$\mathcal{R}_{\text{aux-q}} = \mathcal{R}^{\mathrm{pole}}_{\mathrm{univ}} + a_\epsilon N_c \left(\frac{\mu^2}{|k_T|^2}\right)^{\epsilon} \left\{ \frac{3}{\epsilon} - \frac{2\pi^2}{3} + \frac{7}{2} - \frac{1}{N_c^2}\left[\frac{1}{\epsilon^2} + \frac{3}{2\epsilon} + 4\right]\right\} + \mathcal{O}(\bar{\epsilon}) + \mathcal{O}(\Lambda^{-1}). \tag{275}$$

For auxiliary gluons, we add the contributions from equations (272) and (273), and we also include the number of quark flavors n_f. Hence, we put $\bar{\epsilon} = -2\epsilon$ and $\delta = 1$ to obtain

$$\begin{aligned}\mathcal{R}_{\text{aux-g}} = \mathcal{R}^{\mathrm{pole}}_{\mathrm{univ}} + a_\epsilon N_c \left(\frac{\mu^2}{|k_T|^2}\right)^{\epsilon} \Bigg\{ & \frac{1}{\epsilon^2} + \frac{1}{\epsilon}\frac{11}{2} - \frac{2\pi^2}{3} + \frac{67}{9} \\ & - \frac{n_f}{N_c}\left[\frac{2}{3\epsilon} + \frac{10}{9} - \frac{1}{N_c^2}\left(\frac{1}{3\epsilon} - \frac{1}{6}\right)\right]\Bigg\} + \mathcal{O}(\bar{\epsilon}) + \mathcal{O}(\Lambda^{-1}).\end{aligned} \tag{276}$$

The result for universal term is

$$\mathcal{R}^{\mathrm{pole}}_{\mathrm{univ}} = -\frac{2a_\epsilon N_c}{\epsilon}\left(\frac{\mu^2}{|k_T|^2}\right)^{\epsilon} \ln\frac{\nu^2 C\Lambda}{|k_T|^2}. \tag{277}$$

The unfamiliar virtual contribution calculated in section 4.2 is given by

$$d\mathrm{V}^{\star\mathrm{unf}}\big(k_{\mathrm{in}}, k_{\overline{\mathrm{in}}}; \{p_i\}_n\big) = a_\epsilon N_c \mathrm{Re}(\mathcal{V}_{\mathrm{aux}}) d\mathrm{B}^{\star}\big(k_{\mathrm{in}}, k_{\overline{\mathrm{in}}}; \{p_i\}_n\big). \tag{278}$$

and the $\mathcal{V}_{\mathrm{aux}}$ are exactly given by equations (227) and (228, thus we simply rewrite it here

$$\mathcal{V}_{\text{aux-g}}(h, 1^{\pm}, 2^{\mp}) = \left(\frac{\mu^2}{|k_T|^2}\right)^{\epsilon} \left\{ \frac{2}{\epsilon}\ln\frac{\Lambda}{x} - i\pi - \frac{1}{\epsilon^2} + \frac{\pi^2}{3}\right\} + \mathcal{O}(\epsilon) + \mathcal{O}(\Lambda^{-1}), \tag{279}$$

$$\mathcal{V}_{\text{aux-q}}(h,1^\pm,2^\mp) = \left(\frac{\mu^2}{|k_T|^2}\right)^\epsilon \left\{\frac{2}{\epsilon}\ln\frac{\Lambda}{x} - i\pi + \frac{1}{\epsilon}\frac{13}{6} + \frac{\pi^2}{3} + \frac{80}{18} + \frac{1}{N_c^2}\left[\frac{1}{\epsilon^2} + \frac{3}{2}\frac{1}{\epsilon} + 4\right] - \frac{n_f}{N_c}\left[\frac{2}{3}\frac{1}{\epsilon} + \frac{10}{9}\right]\right\} + \mathcal{O}(\epsilon) + \mathcal{O}(\Lambda^{-1}), \tag{280}$$

where the parameter n_f is the number of flavours running in quark loop.

The completed unfamiliar contribution we find to be

$$dR^{\star\text{unf}} + dV^{\star\text{unf}} = \Delta_{\text{unf}} dB^\star, \tag{281}$$

with

$$\Delta_{\text{unf}} = \frac{a_\epsilon N_c}{\epsilon}\left(\frac{\mu^2}{|k_T|^2}\right)^\epsilon \left[\mathcal{J}_{\text{aux}} + \mathcal{J}_{\text{univ}} + \mathcal{J}_{\text{univ}} - 2\ln\frac{C\nu^2 x}{|k_T|^2}\right], \tag{282}$$

and

$$\mathcal{J}_{\text{univ}} = \frac{11}{6} - \frac{n_f}{3N_c} - \frac{\mathcal{K}}{N_c}(-\epsilon), \quad \mathcal{K} = N_c\left(\frac{67}{18} - \frac{\pi^2}{6}\right) - \frac{5n_f}{9}, \tag{283}$$

$$\mathcal{J}_{\text{aux-q}} = \frac{3}{2} - \frac{1}{2}(-\epsilon), \quad \mathcal{J}_{\text{aux-g}} = \frac{11}{6} + \frac{n_f}{3N_c^3} + \frac{n_f}{6N_c^3}(-\epsilon). \tag{284}$$

Firstly, we would like to stress that the phase space cuts described in sections 5.1 and 2.2, related to Figure 6, only affect the term with the $\ln\Lambda$. The other terms are independent of the choice of the parameters C and D from equations (26) and (27). Nevertheless, the general form of the unfamiliar contribution depends on the scheme used for the finite contribution in the limit $\epsilon \to 0$. It is crucial to include the collinear singularity into the familiar real contribution. As mentioned above, there is arbitrariness in separating the phase space for radiation into the impact factor region and the transverse momentum evolution region. In our work [1], we implemented the scheme proposed and developed in [36], which aims to separate the evolution region from the impact factor region by applying the angular ordering criterion on the radiative gluon. This scheme results in the double IR pole being accounted for the familiar real contribution.

We introduced the quantity $\mathcal{K}$ as it is in equation (4.10) of [35] and we write $(-\epsilon)$ to address their difference in the definition of the parameter for dimensional regularization, which is $\varepsilon = -\epsilon$. We also realize that $\mathcal{J}_{\text{aux-q}} + \mathcal{J}_{\text{univ}}$ is equal to their formula (4.9), while $\mathcal{J}_{\text{aux-g}} + \mathcal{J}_{\text{univ}}$ corresponds to their equation (5.11). The additional $\mathcal{J}_{\text{univ}}$ from equation (282) appears because, as we mentioned in section 4.2, our virtual contribution is not UV-subtracted. By subtracting this term from our result, we can find equations (4.6) and (5.9) in [35].

In our work, the unfamiliar contribution arises from substituting the on-shell gluon in the collinear factorization framework with the space-like gluon introduced through the auxiliary parton method. This originates from the difference between the collinear splitting amplitude of the auxiliary parton into the gluon and the emission amplitude of the space-like t-channel gluon from the auxiliary partons. Thus, it is straightforward that the NLO auxiliary parton impact factor is included in the unfamiliar contribution. Furthermore, within the unfamiliar contribution, we can find terms with $\ln\mu$, which contribute to the off-shell gluon Regge trajectory and form part of the evolution kernel within the high-energy factorization framework. Finally, we must emphasize that the complete unfamiliar contribution does not depend on the type of auxiliary partons, and the $\ln\Lambda$ term is no longer present.

6 General result

The general structure of cross section at NLO within hybrid k_T-factorization we summarize as in [1] to be

$$
\begin{aligned}
d\sigma^{(1)} = \int \frac{dx}{x} d^2k_T \frac{d\bar{x}}{\bar{x}} \\
&\times \Bigg\{ F(x,|k_T|) f(\bar{x}) \Big[d\mathrm{V}^{\star}\big(k_{\mathrm{in}}, k_{\overline{\mathrm{in}}}; \{p_i\}_n\big) + d\mathrm{R}^{\star}\big(k_{\mathrm{in}}, k_{\overline{\mathrm{in}}}; \{p_i\}_{n+1}\big) \Big]_{\text{canceling}} \\
&+ \Big[F^{(1)}(x,|k_T|) + F(x,|k_T|)\Delta_{\mathrm{unf}} + \Delta^{\star}_{\mathrm{coll}} \Big] f(\bar{x}) d\mathrm{B}^{\star}\big(k_{\mathrm{in}}, k_{\overline{\mathrm{in}}}; \{p_i\}_n\big) \\
&+ F(x,|k_T|) \Big[f^{(1)}(\bar{x}) + \Delta_{\overline{\mathrm{coll}}} \Big] d\mathrm{B}^{\star}\big(k_{\mathrm{in}}, k_{\overline{\mathrm{in}}}; \{p_i\}_n\big) \Bigg\},
\end{aligned}
\tag{285}
$$

with the initial states defined in equation (34).

The real contribution term $d\mathrm{R}^{\star}\big(k_{\mathrm{in}}, k_{\overline{\mathrm{in}}}; \{p_i\}_{n+1}\big)$ from the cancelling part requires numerical phase space integration. In section 3 it is explicitly demonstrated how to isolate the divergences within dimensional regularization as $1/\bar{\epsilon}^2$ and $1/\bar{\epsilon}$ contributions. The subscript *canceling* refers to the cancellation between these and those of the virtual contribution. One-loop amplitudes are involved in the virtual contribution $d\mathrm{V}^{\star}\big(k_{\mathrm{in}}, k_{\overline{\mathrm{in}}}; \{p_i\}_n\big)$. In section 4, we introduce off-shell one-loop amplitudes derived from on-shell analytic expressions and described the unfamiliar and familiar contributions. The term Δ_{unf} is discussed in detail in the section 5.2.

The divergent term $\Delta_{\overline{\mathrm{coll}}}$ is discussed in the first part of section 2.2 and is given by

$$
\Delta_{\overline{\mathrm{coll}}} = -\frac{a_\epsilon}{\epsilon} \int_{\bar{x}}^{1} dz \big[\mathcal{P}^{\mathrm{reg}}_{\overline{\mathrm{in}}}(z) + \gamma_{\overline{in}} \delta(1-z) \big] f\left(\frac{\bar{x}}{z}\right). \tag{286}
$$

The existence of $\Delta_{\overline{\mathrm{coll}}}$ is due to the failure of those two contributions to cancel against each other. The $\mathcal{P}^{\mathrm{reg}}_{\overline{\mathrm{in}}}(z)$ part come from equation (150) for the $k_{\overline{\mathrm{in}}}$ case and the $\delta(1-z)$ comes from the familiar virtual contribution.

The other divergence, denoted as $\Delta^{\star}_{\mathrm{coll}}$ is

$$
\Delta^{\star}_{\mathrm{coll}} = -\frac{a_\epsilon}{\epsilon} \int_{x}^{1} dz \left[\frac{2N_c}{[1-z]_+} + \frac{2N_c}{z} + \gamma_g \delta(1-z) \right] F\left(\frac{x}{z}, |k_T|\right). \tag{287}
$$

The first two terms underneath the integral come again form equation (150), but now for the k_{in} case, while the $\delta(1-z)$ is part of the familiar virtual contribution.

7 Conclusions

Within this thesis, we applied the auxiliary parton method to hybrid k_T-factorization and extended it to NLO. We also derived the NLO impact factors, initially obtained in [35]. The main NLO formula for hybrid factorization was addressed in Section 6. In equation (285), we identified both finite and divergent terms. While the real and virtual contributions are individually divergent, some divergences cancel each other out in a manner similar to collinear factorization. We refer to these finite terms as familiar contributions. Additionally, we identified two collinear divergences: the first, $\Delta_{\overline{\text{coll}}}$, cancels against the NLO correction to the corresponding PDF, while the other, $\Delta^{\star}_{\text{coll}}$, is expected to cancel against the transverse momentum-dependent PDF.

The remaining divergent terms form what we call the unfamiliar contribution, as they do not have an equivalent in collinear factorization. These terms were found to have the same structure as the corrections to the impact factors obtained in [35]. This led us to realize that hybrid k_T-factorization must be studied within the framework of high-energy factorization. We also observed that the inclusion of the NLO correction and the Λ limit, as defined in section 1.5, do not commute. Therefore, the Λ limit must be applied afterward, resulting in unfamiliar real or virtual contributions that both depend on $\ln \Lambda$.

We expressed the infrared and ultraviolet divergences in ϵ within dimensional regularization and analyzed their cancellation. The infrared divergences cancel against the NLO target impact factor, while the remaining NLO contributions must be combined with the initial-state parton infrared singularities. For the cancellation of ultraviolet divergences, renormalization must be applied. In the familiar real contribution, we found a connection to the evolution of the k_T-dependent PDF, which was studied in detail in [27]. While expanding on this topic is beyond the scope of this thesis, it is worth noting that exploring this subject in more detail could be valuable for future research.

In section 3, we established the subtraction method for the real NLO correction in hybrid k_T-factorization. Compared to the subtraction schemes for collinear factorization, here we subtract the momentum recoil from the initial-state momenta and modify the final-state momenta in a minimal manner. As expected, we found the divergent term of the integrated subtraction term to be universal. The finite part was presented in a form already prepared for numerical calculations.

We find the real radiation integral to be convergent for all relevant 2-jet processes. Moreover, the final result is independent of the arbitrary parameters introduced for phase space restrictions. The integrated subtraction terms consist of finite parts that can be integrated only numerically over the radiative phase space. This aspect could be found as an significant disadvantage to other subtraction schemes for which the integrated subtraction terms does not demand numerical integration and can be integrated completely analytically over the radiative phase space. However, the extra integrations can be incorporated in the eventual integration over the Born phase space via the Monte Carlo method, and calculating the integrated subtraction terms still takes only a fraction of the time required to calculate the subtracted real contribution.

With this work, we have improved the framework for hybrid k_T-factorization. The results presented in [1, 2] represent a milestone towards the automation of cross section calculations in high energy physics.

A Some spinorology

Throughout the thesis, many formulas, particularly amplitudes, are expressed using spinor notation. Therefore, in this appendix, we would like to introduce the basics of this notation.

The energy and momentum of a relativistic particle can be combined into a four-momentum

$$p = (E, \vec{p}) = (E, p_x, p_y, p_z) = (p_0, p_1, p_2, p_3), \tag{288}$$

with the square

$$p^2 = p_0^2 - p_1^2 - p_2^2 - p_3^2. \tag{289}$$

For light-like momentum p, we can define spinors as four-component vectors

$$|p] = \frac{1}{\sqrt{|p_0 + p_3|}} \begin{pmatrix} -p_1 + ip_2 \\ p_0 + p_3 \\ 0 \\ 0 \end{pmatrix}, \tag{290}$$

$$|p\rangle = \frac{\sqrt{|p_0 + p_3|}}{|p_0 + p_3|} \begin{pmatrix} 0 \\ 0 \\ p_0 + p_3 \\ p_1 + ip_2 \end{pmatrix}, \tag{291}$$

$$[p| = \frac{1}{\sqrt{|p_0 + p_3|}} \Big((p_0 + p_3), (p_1 - ip_2), 0, 0 \Big), \tag{292}$$

$$\langle p| = \frac{\sqrt{|p_0 + p_3|}}{|p_0 + p_3|} \Big(0, 0, (p_1 + ip_2), (-p_0 - p_3) \Big), \tag{293}$$

and we also will use notation for square and angular products

$$\langle p|q\rangle = \langle pq\rangle, \quad [p|q] = [pq]. \tag{294}$$

Within this spinor notation, we can state that the basic objects are the first two $|p]$ and $|p\rangle$, while the remaining two are fixed by these. Spinors follow several useful relations, many of which arise directly from the definitions given above. For example we have

$$[p|q\rangle = \langle p|q] = \langle pp\rangle = [pp] = 0, \tag{295}$$

$$[pq] = -[qp], \quad \langle pq\rangle = -\langle qp\rangle, \tag{296}$$

$$[pq]\langle qp\rangle = 2p\cdot q, \tag{297}$$

$$[pq]^* = \mathrm{sgn}(p_0q_0)\langle qp\rangle. \tag{298}$$

We can also realize that their dyadic products follow the relation

$$[p\rangle[p| + |p]\langle p| = \not{p} = \gamma_\mu p^\mu, \tag{299}$$

where γ matrices are in the Weyl representation. With this relation we can introduce also the following identities

$$2p\cdot q = [p|\not{q}|p\rangle = \langle p|\not{q}|p], \tag{300}$$

$$\not{p}\not{q} = 2p\cdot q - \not{q}\not{p}, \tag{301}$$

$$\langle pq\rangle\langle rs\rangle = \langle ps\rangle\langle rq\rangle + \langle pr\rangle\langle qs\rangle, \tag{302}$$

$$[pq][rs] = [ps][rq] + [pr][qs]. \tag{303}$$

The last two are called the Schouten's identities.

B Triple Λ limit

In this appendix, we introduce the triple Λ limit as a generalization of the auxiliary parton methods, allowing three partons to participate in the consumption of large Λ. We will start with a simple example to illustrate how the method works for five-point partial amplitudes, which can be found, for instance, in [32] and are given by

$$\mathcal{A}(1_q^+, 2_g^+, 3_g^+, 4_g^-, 5_{\bar{q}}^-) = \frac{\langle 14\rangle\langle 54\rangle^3}{\langle 51\rangle\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle}, \tag{304}$$

$$\mathcal{A}(1_q^-, 2_g^-, 3_g^-, 4_g^+, 5_{\bar{q}}^+) = \frac{[41][45]^3}{[54][43][32][21][15]}, \tag{305}$$

please note that the amplitudes are expressed using the spinor notation introduced in appendix A. We also omitted the coupling constant. We write the parametrization of the momenta in the more general notation compared to equation (244), we have

$$k_{\bar{q}}^{\mu} = (x+\Lambda)P^{\mu}, \quad k_q^{\mu} = -x_q\Lambda P^{\mu} + q_T^{\mu} - \frac{q_T^2}{\nu^2 x_q \Lambda}\overline{P}^{\mu}, \quad p_r^{\mu} = -x_r\Lambda P^{\mu} + r_T^{\mu} - \frac{r_T^2}{\nu^2 x_r \Lambda}\overline{P}^{\mu}, \tag{306}$$

with

$$x_r + x_q = 1. \tag{307}$$

Then up to $\mathcal{O}(\Lambda^{-1})$, we have,

$$k_{\bar{q}}^{\mu} + k_q^{\mu} + k_r^{\mu} = xP^{\mu} + q_T^{\mu} + r_T^{\mu}, \quad (k_{\bar{q}} + k_q + k_r)^2 = (q_T + r_T)^2, \tag{308}$$

and

$$s_{q\bar{q}} = (k_{\bar{q}} + k_q)^2 = \frac{q_T^2}{x_q}, \tag{309}$$

$$s_{r\bar{q}} = (k_{\bar{q}} + k_r)^2 = \frac{r_T^2}{x_r}, \tag{310}$$

$$s_{rq} = (k_r + k_q)^2 = (q_T + r_T)^2 - \frac{q_T^2}{x_q} - \frac{r_T^2}{x_r}, \tag{311}$$

The spinors are parametrized as follow

$$|i_{\bar{q}}\rangle = \sqrt{x+\Lambda}|P\rangle, \quad |i_{\bar{q}}] = -\sqrt{x+\Lambda}|P], \tag{312}$$

$$|j_q\rangle = \sqrt{x_q\Lambda}|P\rangle + \frac{\kappa_q^*}{\sqrt{x_q\Lambda}\langle P\overline{P}\rangle}|\overline{P}\rangle, \quad |j_q] = \sqrt{x_q\Lambda}|P] + \frac{\kappa_q}{\sqrt{x_q\Lambda}[\overline{P}]}|\overline{P}], \tag{313}$$

$$|l_r\rangle = \sqrt{x_r\Lambda}|P\rangle + \frac{\kappa_r^*}{\sqrt{x_r\Lambda}\langle P\overline{P}\rangle}|\overline{P}\rangle, \quad |l_r] = \sqrt{x_r\Lambda}|P] + \frac{\kappa_r}{\sqrt{x_r\Lambda}[\overline{P}]}|\overline{P}], \tag{314}$$

with products

$$\langle j_q i_{\bar{q}}\rangle = -\frac{\kappa_q^*}{\sqrt{x_q}}, \quad [i_{\bar{q}} j_q] = \frac{\kappa_q}{\sqrt{x_q}}, \tag{315}$$

$$\langle l_r i_{\bar{q}}\rangle = -\frac{\kappa_r^*}{\sqrt{x_r}}, \quad [i_{\bar{q}} l_r] = \frac{\kappa_r}{\sqrt{x_r}}, \tag{316}$$

$$\langle l_r j_q\rangle = \sqrt{\frac{x_r}{x_q}}\kappa_q^* - \sqrt{\frac{x_q}{x_r}}\kappa_r^*, \quad [j_q l_r] = \sqrt{\frac{x_r}{x_q}}\kappa_q - \sqrt{\frac{x_q}{x_r}}\kappa_r, \tag{317}$$

and κ's defined as

$$\kappa_r\kappa_r^* = -r_T^2, \quad \kappa_q\kappa_q^* = -q_T^2. \tag{318}$$

We apply this to our example amplitudes given by equations (304) and (305) with the choice of auxiliary partons to be $i = 5$, $j = 1$ and $l = 2$. For the firs one we have

$$\begin{aligned}
&\mathcal{A}(1_q^+, 2_g^+, 3_g^+, 4_g^-, 5_{\overline{q}}^-) = \frac{\langle 14\rangle\langle 54\rangle^3}{\langle 51\rangle\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle} \\
&\to \frac{\sqrt{x_q\Lambda}\langle P4\rangle(\sqrt{\Lambda})^3\langle P4\rangle^3}{\langle 51\rangle\langle 12\rangle\sqrt{x_r\Lambda}\langle P3\rangle\langle 34\rangle\sqrt{\Lambda}\langle 4P\rangle} = \sqrt{\frac{x_q}{x_r}}\frac{\langle 25\rangle}{\langle 51\rangle\langle 12\rangle\langle 25\rangle}\frac{\Lambda\langle P4\rangle^4}{\langle P3\rangle\langle 34\rangle\langle 4P\rangle} \\
&= \sqrt{\frac{x_q}{x_r}}\frac{(\kappa_q^* + \kappa_r^*)\langle 25\rangle}{\langle 51\rangle\langle 12\rangle\langle 25\rangle}\frac{\Lambda\langle P4\rangle^4}{(\kappa_q^* + \kappa_r^*)\langle P3\rangle\langle 34\rangle\langle 4P\rangle} = \sqrt{\frac{x_q}{x_r}}\frac{(\kappa_q^* + \kappa_r^*)\langle 25\rangle}{\langle 51\rangle\langle 12\rangle\langle 25\rangle}\Lambda\mathcal{A}(g^\star, 3_g^+, 4_g^-),
\end{aligned} \tag{319}$$

and for the second

$$\begin{aligned}
&\mathcal{A}(1_q^-, 2_g^-, 3_g^-, 4_g^+, 5_{\overline{q}}^+) = \frac{[41][45]^3}{[54][43][32][21][15]} \\
&\to \frac{\sqrt{x_q\Lambda}[4P](\sqrt{\Lambda})^3[4P]^3}{\sqrt{\Lambda}[P4][43][3P]\sqrt{x_r\Lambda}[21][15]} = \sqrt{\frac{x_q}{x_r}}\frac{[25]}{[51][12][25]}\frac{\Lambda[4P]^4}{[P4][43][3P]} \\
&= \sqrt{\frac{x_q}{x_r}}\frac{(\kappa_q + \kappa_r)[25]}{[51][12][25]}\frac{\Lambda[4P]^4}{(\kappa_q + \kappa_r)[P4][43][3P]} = \sqrt{\frac{x_q}{x_r}}\frac{(\kappa_q + \kappa_r)[25]}{[51][12][25]}\Lambda\mathcal{A}(g^\star, 3_g^-, 4_g^+).
\end{aligned} \tag{320}$$

We introduce the following functions for simplification of the notation of the so called "splitting functions"

$$\mathrm{Sqr}(\overline{q}, q, r) = \frac{\kappa_q + \kappa_r}{[i_{\overline{q}}j_q][j_ql_r][l_ri_{\overline{q}}]}, \quad \mathrm{Ang}(\overline{q}, q, r) = \frac{\kappa_q^* + \kappa_r^*}{\langle i_{\overline{q}}j_q\rangle\langle j_ql_r\rangle\langle l_ri_{\overline{q}}\rangle}. \tag{321}$$

B.1 Splitting functions for auxiliary quark pair and a gluon

Triple Λ limit produce the following structure

$$\begin{aligned}
\mathcal{A}(1_q, 2_r, \cdots, n_{\overline{q}}) &\to \Lambda\mathcal{S}(\overline{q}, q, r)\mathcal{A}^\star(g^\star, \cdots) \\
\mathcal{A}(1_q, \cdots, (n-1)_r, n_{\overline{q}}) &\to \Lambda\mathcal{S}(r, \overline{q}, q)\mathcal{A}^\star(g^\star, \cdots).
\end{aligned} \tag{322}$$

The functions $\mathcal{S}$, for the helicities corresponding to non-vanishing amplitudes, are given by

$$
\begin{aligned}
&\mathcal{S}(\overline{q}^+,q^-,r^-)=\mathrm{Sqr}(\overline{q},q,r)\frac{\sqrt{x_q}[l_r i_{\overline{q}}]}{\sqrt{x_r}}, \quad \mathcal{S}(\overline{q}^-,q^+,r^+)=\mathrm{Ang}(\overline{q},q,r)\frac{\sqrt{x_q}\langle l_r i_{\overline{q}}\rangle}{\sqrt{x_r}}\\
&\mathcal{S}(\overline{q}^+,q^-,r^+)=\mathrm{Ang}(\overline{q},q,r)\frac{x_q^{3/2}\langle l_r i_{\overline{q}}\rangle}{\sqrt{x_r}}, \quad \mathcal{S}(\overline{q}^-,q^+,r^-)=\mathrm{Sqr}(\overline{q},q,r)\frac{x_q^{3/2}[l_r i_{\overline{q}}]}{\sqrt{x_r}}\\
&\mathcal{S}(r^-,\overline{q}^+,q^-)=\mathrm{Sqr}(\overline{q},q,r)\frac{-[j_q l_r]}{\sqrt{x_r}}, \quad \mathcal{S}(r^+,\overline{q}^-,q^+)=\mathrm{Ang}(\overline{q},q,r)\frac{\langle j_q l_r\rangle}{\sqrt{x_r}}\\
&\mathcal{S}(r^+,\overline{q}^+,q^-)=\mathrm{Ang}(\overline{q},q,r)\frac{x_q\langle j_q l_r\rangle}{\sqrt{x_r}}, \quad \mathcal{S}(r^-,\overline{q}^-,q^+)=\mathrm{Sqr}(\overline{q},q,r)\frac{-x_q[j_q l_r]}{\sqrt{x_r}}.
\end{aligned}
\tag{323}
$$

B.2 Splitting functions for auxiliary gluons

Now the labels q and $\overline{q}$ refer to gluon pair, but we keep them for convenience. We have the same structure as in equation (322). The functions $\mathcal{S}$, for non-vanishing amplitudes, are given by

$$
\begin{aligned}
\mathcal{S}(\overline{q}^+,q^-,r^-)&=\mathrm{Sqr}(\overline{q},q,r)\frac{-\sqrt{x_q}[l_r i_{\overline{q}}]}{\sqrt{x_q x_r}}\\
\mathcal{S}(\overline{q}^+,q^-,r^+)&=\mathrm{Ang}(\overline{q},q,r)\frac{x_q^{5/2}\langle l_r i_{\overline{q}}\rangle}{\sqrt{x_q x_r}}\\
\mathcal{S}(\overline{q}^+,q^+,r^-)&=\mathrm{Ang}(\overline{q},q,r)\frac{\sqrt{x_q}x_r^2\langle l_r i_{\overline{q}}\rangle}{\sqrt{x_q x_r}}\\
\mathcal{S}(r^-,\overline{q}^+,q^-)&=\mathrm{Sqr}(\overline{q},q,r)\frac{[j_q l_r]}{\sqrt{x_q x_r}}\\
\mathcal{S}(r^+,\overline{q}^+,q^-)&=\mathrm{Ang}(\overline{q},q,r)\frac{x_q^2\langle j_q l_r\rangle}{\sqrt{x_q x_r}}\\
\mathcal{S}(r^-,\overline{q}^+,q^+)&=\mathrm{Ang}(\overline{q},q,r)\frac{x_r^2\langle j_q l_r\rangle}{\sqrt{x_q x_r}}.
\end{aligned}
\tag{324}
$$

B.3 Splitting functions for auxiliary initial gluon and final quark pair

In this case, there is a slight modification in the parametrization for $|l_r]$, as it is now in the initial state. Therefore, there is an exchange of the labels $\overline{q}$ and r within equations (312) and (314). The splitting functions are

$$
\begin{aligned}
\mathcal{S}_g(\overline{q}^+, q^-, r^-) &= \mathrm{Sqr}(\overline{q}, q, r)(-\sqrt{x_q}x_{\overline{q}}[l_r i_{\overline{q}}]) \\
\mathcal{S}_g(\overline{q}^+, q^-, r^+) &= \mathrm{Ang}(\overline{q}, q, r)(x_q^{3/2}\langle l_r i_{\overline{q}}\rangle) \\
\mathcal{S}_g(\overline{q}^-, q^+, r^+) &= \mathrm{Ang}(\overline{q}, q, r)(\sqrt{x_q}x_{\overline{q}}\langle l_r i_{\overline{q}}\rangle) \\
\mathcal{S}_g(\overline{q}^-, q^+, r^-) &= \mathrm{Sqr}(\overline{q}, q, r)(-x_q^{3/2}[l_r i_{\overline{q}}]) \\
\mathcal{S}_g(r^-, \overline{q}^+, q^-) &= \mathrm{Sqr}(\overline{q}, q, r)(-x_{\overline{q}}^{3/2}[j_q l_r]) \\
\mathcal{S}_g(r^+, \overline{q}^+, q^-) &= \mathrm{Ang}(\overline{q}, q, r)(x_q\sqrt{x_{\overline{q}}}\langle j_q l_r\rangle) \\
\mathcal{S}_g(r^+, \overline{q}^-, q^+) &= \mathrm{Ang}(\overline{q}, q, r)(x_{\overline{q}}^{3/2}\langle j_q l_r\rangle) \\
\mathcal{S}_g(r^-, \overline{q}^-, q^+) &= \mathrm{Sqr}(\overline{q}, q, r)(-x_q\sqrt{x_{\overline{q}}}[j_q l_r]).
\end{aligned} \tag{325}
$$

B.4 Squared matrix element

Any amplitudes for a quark-antiquark pair and $n+1$ gluons can be decomposed for any helicity configuration as follows

$$
\mathcal{M}_{\text{tree}}^{a_1 a_2 \cdots a_{n+1} j i} = 2^{\frac{n+1}{2}} \sum_{\sigma \in S_{n+1}} Tr\left(T^{a_{\sigma(1)}} \ldots T^{a_{\sigma(n+1)}}\right)_{ji} \mathcal{A}\left(0_q, \sigma(1)_g, \ldots, \sigma(n+1)_{g_r}, (n+2)_{\overline{q}}\right), \tag{326}
$$

where $\mathcal{A}$ is the color-ordered partial amplitude and the $n+1$ gluon is the radiative gluon. We realize that the amplitudes contributing to the large Λ limit are those where the radiative gluon is the first or the last one. Then we obtain

$$
\begin{aligned}
\mathcal{M}_{\text{tree}}^{a_1 a_2 \cdots a_n a_r j i} \to \Lambda 2^{\frac{n+1}{2}} \Bigg\{ &\mathcal{S}(\overline{q}, q, r) \sum_{\sigma \in S_n} Tr\left(T^{a_r} T^{a_{\sigma(1)}} \ldots T^{a_{\sigma(n)}}\right)_{ji} \mathcal{A}\left(g^\star, \sigma(1)_g, \ldots, \sigma(n)_g\right) \\
&+ \mathcal{S}(r, \overline{q}, q) \sum_{\sigma \in S_n} Tr\left(T^{a_{\sigma(1)}} \ldots T^{a_{\sigma(n)}} T^{a_r}\right)_{ji} \mathcal{A}\left(g^\star, \sigma(1)_g, \ldots, \sigma(n)_g\right) \Bigg\}.
\end{aligned} \tag{327}
$$

This has to be squared, then summed over color and helicities to find the complete formula for the squared matrix element. Thus, we get

$$
\begin{aligned}
&|\overline{\mathcal{M}}|^2(\Lambda + x, \overline{x}; k_r, k_q, \{p_i\}_n) \\
&\qquad \xrightarrow{\Lambda \to \infty} C_{\text{aux}} \mathcal{Q}_{\text{aux}}(x_q, q_T, x_r, q_T) \frac{\Lambda^2 |\overline{\mathcal{M}}^\star|^2 \big(xP + q_T + r_T, \overline{x}; \{p_i\}_n\big)}{x^2 |q_T + r_T|^2},
\end{aligned} \tag{328}
$$

with

$$\mathcal{Q}_{\text{aux}}(x_q, q_T, x_r, q_T) = \mathcal{P}_{\text{aux}}(x_q, x_r)|q_T + r_T|^2 \left(\frac{c_{\overline{q}}}{|s_{r\overline{q}}||s_{q\overline{q}}|} + \frac{c_q x_q}{|s_{rq}||s_{q\overline{q}}|} + \frac{c_r x_r}{|s_{rq}||s_{r\overline{q}}|} \right). \tag{329}$$

Applying equations (309), (310) and (311) we will obtain equation (249). The other coefficients for auxiliary partons are

$$\text{auxiliary quarks:} \quad \begin{aligned} &\mathcal{P}_{\text{aux}}(x_q, x_r) = \frac{1 + x_q^2 - \epsilon x_r^2}{x_r}, \\ &C_{\text{aux}} = \frac{N_c^2 - 1}{N_c}, \quad c_q = c_{\overline{q}} = N_c, \quad c_r = \frac{-1}{N_c}, \end{aligned} \tag{330}$$

$$\text{auxiliary gluons:} \quad \begin{aligned} &\mathcal{P}_{\text{aux}}(x_q, x_r) = \frac{1 + x_q^4 + x_r^4}{x_q x_r}, \\ &C_{\text{aux}} = 2N_c, \quad c_q = c_{\overline{q}} = c_r = N_c. \end{aligned} \tag{331}$$

For initial state gluon and final state quark-antiquark pair we have to exchange $r \leftrightarrow \overline{q}$ in equation (329) to have

$$\mathcal{Q}_{\text{aux}}(x_q, q_T, x_r, q_T) = \mathcal{P}_{\text{aux}}(x_q, x_r)|q_T + \overline{q}_T|^2 \left(\frac{c_r}{|s_{r\overline{q}}||s_{qr}|} + \frac{c_q x_q}{|s_{rq}||s_{q\overline{q}}|} + \frac{c_{\overline{q}} x_{\overline{q}}}{|s_{\overline{q}q}||s_{\overline{q}r}|} \right), \tag{332}$$

and

$$\begin{aligned} &\mathcal{P}_{\text{aux}}(x_q, x_r) = 1 - \frac{2x_q x_r}{1 - \epsilon}, \\ &C_{\text{aux}} = \frac{N_c^2 - 1}{N_c} \quad c_q = c_{\overline{q}} = \frac{1}{2}, \quad c_r = \frac{-1}{2N_c^2}. \end{aligned} \tag{333}$$

All calculations of the splitting functions were performed with the help of the FORM program [37]. We also used the explicit formulas for the squared matrix elements from [38] to cross-check the terms proportional to ϵ from dimensional regularization.

C Calculations for $\mathcal{R}_{\text{aux}}^{\text{rest}}$

We would like to start with the decomposition of $\mathcal{P}_{\text{aux}}(x_q, x_r)$ given by equation (263). Inserting the relation given by equation (245) for x_q into the formulas for auxiliary quarks (251), gluons (252) and gluon to quarks (253), we obtain

$$\text{for auxiliary quarks:} \quad \mathcal{P}_{\text{aux}}(x_r) = \frac{2}{x_r} - 2 + x_r + \frac{1}{2}\delta\overline{\epsilon}x_r = \frac{\mathcal{P}_{\text{aux}}^{\text{pole}}}{x_r} + \mathcal{P}_{\text{aux-q}}^{\text{rest}}(x_r), \tag{334}$$

$$\text{for auxiliary gluons:}\quad \mathcal{P}_{\text{aux}}(x_r) = \frac{2}{x_r} - 3 + 3x_r - x_r^2 + \frac{1+x_r^3}{1-x_r} = \frac{\mathcal{P}_{\text{aux}}^{\text{pole}}}{x_r} + \mathcal{P}_{\text{aux-g}}^{\text{rest}}(x_r), \tag{335}$$

$$\text{gluon to quarks:}\quad \mathcal{P}_{\text{aux}}(x_r) = 1 - \frac{2(1-x_r)x_r}{1+\frac{1}{2}\delta\bar{\epsilon}} = \mathcal{P}_{\text{aux-gq}}^{\text{rest}}(x_r), \tag{336}$$

the δ was introduced for latter convenience. We realize that indeed $\mathcal{P}_{\text{aux}}^{\text{pole}}$ is independent of the type of auxiliary partons and equals 2, while the $\mathcal{P}_{\text{aux}}^{\text{rest}}(x_r)$ parts differ.

The $\mathcal{R}_{\text{aux}}^{\text{rest}}$ from the decomposition given by equation (265), after taking the limit $\Lambda \to \infty$ and then performing the $\overline{x}_r$ integration, is given by

$$\begin{aligned}
\mathcal{R}_{\text{aux}}^{\text{rest}} =& \frac{a_\epsilon |k_T|^2}{2\pi_\epsilon \mu^{\bar{\epsilon}}} \\
&\times \left[\int_0^{x_C} dx_r \int d^{2+\bar{\epsilon}} r_T \theta(0 < |r_T|^2 < \infty) \right. \\
&\qquad \left. + \int_{x_C}^{u} dx_r \int d^{2+\bar{\epsilon}} r_T \theta(0 < |r_T|^2 < \infty) \right] \\
&\times \left\{ \frac{\mathcal{P}_{\text{aux}}^{\text{pole}}}{x_r} c_q (T_q - T_{\bar{q}}) + \frac{\mathcal{P}_{\text{aux}}^{\text{pole}}}{x_r} c_r T_r + \mathcal{P}_{\text{aux}}^{\text{rest}}(x_r) [c_{\bar{q}} T_{\bar{q}} + c_q T_q + c_r T_r] \right\} \\
&- \frac{a_\epsilon |k_T|^2}{2\pi_\epsilon \mu^{\bar{\epsilon}}} \\
&\times \left[\int_u^{x_C} dx_r \int d^{2+\bar{\epsilon}} r_T \theta(0 < |r_T|^2 < \infty) \right. \\
&\qquad \left. + \int_{x_C}^{1} dx_r \int d^{2+\bar{\epsilon}} r_T \theta(0 < |r_T|^2 < \infty) \right] \\
&\times \frac{\mathcal{P}_{\text{aux}}^{\text{pole}}}{x_r} (c_{\bar{q}} + c_q) T_{\bar{q}}.
\end{aligned} \tag{337}$$

After reorganization we obtain

$$\begin{aligned}
\mathcal{R}_{\text{aux}}^{\text{rest}} =& \frac{a_\epsilon |k_T|^2}{2\pi_\epsilon \mu^{\bar{\epsilon}}} \int_0^{u} dx_r \int d^{2+\bar{\epsilon}} r_T \theta(0 < |r_T|^2 < \infty) \\
&\times \left\{ \frac{\mathcal{P}_{\text{aux}}^{\text{pole}}}{x_r} c_q (T_q - T_{\bar{q}}) + \frac{\mathcal{P}_{\text{aux}}^{\text{pole}}}{x_r} c_r T_r + \mathcal{P}_{\text{aux}}^{\text{rest}}(x_r) [c_{\bar{q}} T_{\bar{q}} + c_q T_q + c_r T_r] \right\} \\
&- \frac{a_\epsilon |k_T|^2}{2\pi_\epsilon \mu^{\bar{\epsilon}}} \int_u^{1} dx_r \int d^{2+\bar{\epsilon}} r_T \theta(0 < |r_T|^2 < \infty) \\
&\times \frac{\mathcal{P}_{\text{aux}}^{\text{pole}}}{x_r} (c_{\bar{q}} + c_q) T_{\bar{q}},
\end{aligned} \tag{338}$$

with T_i defined in equations (260), (261), and (262). Considering the five terms in braces, we do the following decomposition

$$\mathcal{R}_{\text{aux}}^{\text{rest}} = \mathcal{R}_{\text{aux}}^{\text{rest},1} + \mathcal{R}_{\text{aux}}^{\text{rest},2} + \mathcal{R}_{\text{aux}}^{\text{rest},3} + \mathcal{R}_{\text{aux}}^{\text{rest},4}. \tag{339}$$

For the $\mathcal{R}_{\text{aux}}^{\text{rest},1}$ we have the following expression

$$\begin{aligned}\mathcal{R}_{\text{aux}}^{\text{rest},1} =& \frac{a_\epsilon |k_T|^2}{2\pi_\epsilon \mu^{\bar{\epsilon}}} \int_0^u dx_r \int d^{2+\bar{\epsilon}} r_T \theta(0 < |r_T|^2 < \infty) \\ &\times \frac{2}{x_r} c_q \bigg(\frac{(1-x_r)^2}{|r_T + k_T|^2 |r_T + x_r k_T|^2} - \frac{1}{|r_T|^2 |r_T + k_T|^2} \bigg),\end{aligned} \tag{340}$$

shifting the $r_T \to r_T - k_T$ we find

$$\begin{aligned}\mathcal{R}_{\text{aux}}^{\text{rest},1} &= \frac{2a_\epsilon}{\bar{\epsilon}} \bigg(\frac{|k_T|}{\mu} \bigg)^{\bar{\epsilon}} N_c \int_0^u dx_r \frac{2}{x_r} \big[(1-x_r)^{\bar{\epsilon}} - 1 \big] \\ &= 4a_\epsilon \bigg(\frac{|k_T|}{\mu} \bigg)^{\bar{\epsilon}} N_c \big[-\operatorname{Li}_2(u) + \mathcal{O}(\bar{\epsilon}) \big].\end{aligned} \tag{341}$$

We can realize that that for $\mathcal{R}_{\text{aux}}^{\text{rest},2}$, there is a possibility of scaling r_T with x_r, allowing us to write

$$\mathcal{R}_{\text{aux}}^{\text{rest},2} = \frac{a_\epsilon |k_T|^2}{2\pi_\epsilon \mu^{\bar{\epsilon}}} c_r \int_0^u dx_r \frac{2}{x_r} x_r^{\bar{\epsilon}} \int d^{2+\bar{\epsilon}} r_T \frac{\theta(0 < |r_T|^2 < \infty)}{|r_T|^2 |r_T + k_T|^2}. \tag{342}$$

We observe that the $1/x_r$ singularity is protected by $\bar{\epsilon}$. Thanks to this, we can safely take the limit $\Lambda \to \infty$. Calculating the integral with the help of equation (364), we obtain

$$\mathcal{R}_{\text{aux}}^{\text{rest},2} = \frac{4a_\epsilon}{\bar{\epsilon}^2} \bigg(\frac{|k_T|}{\mu} \bigg)^{\bar{\epsilon}} c_r u^{\bar{\epsilon}}. \tag{343}$$

For the third component, we can apply equation (364), we obtain

$$\begin{aligned}\mathcal{R}_{\text{aux}}^{\text{rest},3} =& \frac{a_\epsilon |k_T|^2}{2\pi_\epsilon \mu^{\bar{\epsilon}}} \int_0^u dx_r \mathcal{P}_{\text{aux}}^{\text{rest}}(x_r) \int d^{2+\bar{\epsilon}} r_T \theta(0 < |r_T|^2 < \infty) \\ &\times \left[\frac{c_{\bar{q}}}{|r_T|^2 |r_T + k_T|^2} + \frac{c_q (1-x_r)^2}{|r_T + k_T|^2 |r_T + x k_T|^2} + \frac{c_r x_r^2}{|r_T|^2 |r_T + x k_T|^2} \right] \\ =& \frac{2a_\epsilon}{\bar{\epsilon}} \bigg(\frac{|k_T|}{\mu} \bigg)^{\bar{\epsilon}} \int_0^u dx_r \mathcal{P}_{\text{aux}}^{\text{rest}}(x_r) [c_{\bar{q}} + c_q (1-x_r)^{\bar{\epsilon}} + c_r x^{\bar{\epsilon}}].\end{aligned} \tag{344}$$

The result obviously varies for distinct kinds of auxiliary partons. This variation arises from the different forms of $\mathcal{P}_{\text{aux}}^{\text{rest}}(x_r)$ given in equations (334), (335), and (336).

The fourth component is given by

$$
\begin{aligned}
\mathcal{R}^{\text{rest},4}_{\text{aux}} =& -\frac{a_\epsilon |k_T|^2}{2\pi_\epsilon \mu^{\bar\epsilon}} \int_u^1 dx_r \int d^{2+\bar\epsilon} r_T \theta(0<|r_T|^2<\infty) \\
&\times \frac{2}{x_r} \frac{c_{\bar q}+c_q}{|r_T|^2 |r_T+k_T|^2},
\end{aligned} \tag{345}
$$

again with equation (364) we have

$$
\mathcal{R}^{\text{rest},4}_{\text{aux}} = -\frac{4a_\epsilon}{\bar\epsilon}\left(\frac{|k_T|}{\mu}\right)^{\bar\epsilon} \int_u^1 dx_r \frac{(c_{\bar q}+c_q)}{x_r}. \tag{346}
$$

Performing x_r integral and putting the coefficients $c_{\bar q} = c_q = N_c$ we obtain

$$
\mathcal{R}^{\text{rest},4}_{\text{aux}} = \frac{8a_\epsilon N_c}{\bar\epsilon}\left(\frac{|k_T|}{\mu}\right)^{\bar\epsilon} \ln u. \tag{347}
$$

For auxiliary quarks we have

$$
\mathcal{R}^{\text{rest},3}_{\text{aux-q}} = \frac{2a_\epsilon}{\bar\epsilon}\left(\frac{|k_T|}{\mu}\right)^{\bar\epsilon} \int_0^u dx_r \left[-2+x_r+\frac{1}{2}\delta\bar\epsilon x_r\right]\left[c_{\bar q}+c_q(1-x_r)^{\bar\epsilon}+c_r x^{\bar\epsilon}\right], \tag{348}
$$

putting $u=1$ and doing the x_r integral

$$
\mathcal{R}^{\text{rest},3}_{\text{aux-q}} = \frac{2a_\epsilon}{\bar\epsilon}\left(\frac{|k_T|}{\mu}\right)^{\bar\epsilon} \left\{[c_{\bar q}+c_q+c_r]\left(-\frac{3}{2}+\frac{\delta}{4}\bar\epsilon\right)+c_q\frac{5}{4}\bar\epsilon+c_r\frac{7}{4}\bar\epsilon+\mathcal{O}(\bar\epsilon^2)\right\}. \tag{349}
$$

We can neglect the $\mathcal{O}(\bar\epsilon^2)$ and put the coefficients $c_{\bar q} = c_q = N_c$, $c_r = -1/N_c$, then we obtain

$$
\mathcal{R}^{\text{rest},3}_{\text{aux-q}} = 2a_\epsilon N_c \left(\frac{|k_T|}{\mu}\right)^{\bar\epsilon} \left\{-\frac{3}{\bar\epsilon}+\frac{5+2\delta}{4}-\frac{1}{N_c^2}\left[-\frac{3}{2\bar\epsilon}+\frac{7+\delta}{4}\right]\right\}. \tag{350}
$$

For $u=1$ from equation (347) we have $\mathcal{R}^{\text{rest},4}_{\text{aux-q}} = 0$. Collecting all with the equations (341) and (343) and having in mind that $\text{Li}_2(1) = \pi^2/6$ we have

$$
\mathcal{R}^{\text{rest}}_{\text{aux-q}} = 2a_\epsilon N_c \left(\frac{|k_T|}{\mu}\right)^{\bar\epsilon} \left\{-\frac{3}{\bar\epsilon}-\frac{\pi^2}{3}+\frac{5+2\delta}{4}-\frac{1}{N_c^2}\left[\frac{2}{\bar\epsilon^2}-\frac{3}{2\bar\epsilon}+\frac{7+\delta}{4}\right]\right\}. \tag{351}
$$

For auxiliary gluons we have

$$
\mathcal{R}^{\text{rest},3}_{\text{aux-g}} = \frac{2a_\epsilon}{\bar\epsilon}\left(\frac{|k_T|}{\mu}\right)^{\bar\epsilon} \int_0^u dx_r \times 2\left[\frac{1}{1-x_r}-2+x_r-x_r^2\right]\left[c_{\bar q}+c_q(1-x_r)^{\bar\epsilon}+c_r x^{\bar\epsilon}\right], \tag{352}
$$

putting $u=1/2$ and performing the x_r integral

$$\begin{aligned}\mathcal{R}^{\text{rest},3}_{\text{aux-g}} =&\frac{2a_\epsilon}{\bar\epsilon}\left(\frac{|k_T|}{\mu}\right)^{\bar\epsilon}\Bigg\{[c_{\bar q}+c_q+c_r]\left(2\ln 2-\frac{11}{6}\right)+c_q\left(\frac{131}{72}-\frac{132}{72}\ln 2-\ln^2 2\right)\bar\epsilon \\ &+c_r\left(\frac{137}{72}+\frac{132}{72}\ln 2-\ln^2 2-\frac{\pi^2}{6}\right)\bar\epsilon+\mathcal{O}(\bar\epsilon^2)\Bigg\}.\end{aligned} \tag{353}$$

Again we neglect the $\mathcal{O}(\bar\epsilon^2)$ and put the coefficients $c_{\bar q}=c_q=c_r=N_c$, then we have

$$\mathcal{R}^{\text{rest},3}_{\text{aux-g}} = 2a_\epsilon N_c\left(\frac{|k_T|}{\mu}\right)^{\bar\epsilon}\left\{\frac{3}{\bar\epsilon}\left(2\ln 2-\frac{11}{6}\right)+\frac{67}{18}-\frac{\pi^2}{6}-2\ln^2 2\right\}. \tag{354}$$

Now for $u=1/2$ equation (347) gives

$$\mathcal{R}^{\text{rest},4}_{\text{aux-g}} = -\frac{8a_\epsilon N_c}{\bar\epsilon}\left(\frac{|k_T|}{\mu}\right)^{\bar\epsilon}\ln 2. \tag{355}$$

Adding the result from equations (341), (343), (354 and (355) and having in mind that

$$\text{Li}_2\left(\frac{1}{2}\right)=\frac{\pi^2}{12}-\frac{\ln^2 2}{2}, \tag{356}$$

we have

$$\mathcal{R}^{\text{rest}}_{\text{aux-g}} = 2a_\epsilon N_c\left(\frac{|k_T|}{\mu}\right)^{\bar\epsilon}\left\{\frac{2}{\bar\epsilon^2}-\frac{1}{\bar\epsilon}\frac{11}{2}-\frac{\pi^2}{3}+\frac{67}{18}\right\}. \tag{357}$$

For initial state gluon and final state quark pair $\mathcal{R}^{\text{rest}}_{\text{aux-gq}}=\mathcal{R}^{\text{rest},3}_{\text{aux-gq}}$

$$\begin{aligned}\mathcal{R}^{\text{rest}}_{\text{aux-gq}} =&\frac{2a_\epsilon}{\bar\epsilon}\left(\frac{|k_T|}{\mu}\right)^{\bar\epsilon} \\ &\times\int_0^u dx_r\Big[1-2x_r+2x_r^2+(x_r-x_r^2)\delta\bar\epsilon+\mathcal{O}(\bar\epsilon^2)\Big]\Big[c_{\bar q}+c_q(1-x_r)^{\bar\epsilon}+c_r x^{\bar\epsilon}\Big],\end{aligned} \tag{358}$$

with $u=1$ the x_r integral gives

$$\mathcal{R}^{\text{rest}}_{\text{aux-gq}} =\frac{2a_\epsilon}{\bar\epsilon}\left(\frac{|k_T|}{\mu}\right)^{\bar\epsilon}\left\{[c_{\bar q}+c_q+c_r]\left(\frac{2}{3}+\frac{\delta}{6}\bar\epsilon\right)-[c_q+c_r]\frac{13}{18}\bar\epsilon+\mathcal{O}(\bar\epsilon^2)\right\}. \tag{359}$$

We rather keep the labels r and q for the final state partons, in this case the quark pair. With that, we put the coefficients $c_{\bar q}=-1/(2N_c^2)$ and $c_q=c_r=1/2$. Then we find

$$\mathcal{R}^{\text{rest}}_{\text{aux-gq}} =2a_\epsilon\left(\frac{|k_T|}{\mu}\right)^{\bar\epsilon}\left\{\frac{2}{3\bar\epsilon}-\frac{13}{18}+\frac{\delta}{6}-\frac{1}{N_c^2}\left(\frac{1}{3\bar\epsilon}+\frac{\delta}{12}\right)\right\}. \tag{360}$$

D Color correlated matrix elements

In this appendix we would like to describe the color correlated matrix elements. The sum over color we write within color indices as $\mathcal{M}^{\omega_1\omega_2\cdots\omega_n}$ where ω_l is the adjoint index a_l if l refers to a gluon, it is the fundamental index i_l when l refers to a quark and it is the anti-fundamental index j_l when l refers to an anti-quark. The squared matrix element summed over color is given by

$$|\mathcal{M}|^2 = \left(\mathcal{M}^{\overline{\omega}_1\overline{\omega}_2\cdots\overline{\omega}_n}\right)^* \delta_{\overline{\omega}_1\omega_1}\delta_{\overline{\omega}_2\omega_2}\cdots\delta_{\overline{\omega}_n\omega_n}\mathcal{M}^{\omega_1\omega_2\cdots\omega_n}. \tag{361}$$

where the usual summation over repeated index indices is applied. The color-correlated squared sum $\Big(\mathcal{M}\Big)^2_{\text{color}(k,l)}$ is obtained by the following substitution

$$\delta_{\overline{\omega}_k\omega_k}\delta_{\overline{\omega}_l\omega_l} \to \mathcal{T}^c_{\overline{\omega}_k\omega_k}\mathcal{T}^c_{\overline{\omega}_l\omega_l}, \tag{362}$$

where

$$\mathcal{T}^c_{\overline{\omega}_l\omega_l} = \begin{cases} \mathrm{i}f^{\overline{a}_l c a_l} & \text{if } l \text{ refers to a gluon,} \\ T^c_{\overline{i}_l i_l} & \text{if } l \text{ refers to a quark,} \\ -T^c_{j_l\overline{j}_l} & \text{if } l \text{ refers to an anti-quark.} \end{cases} \tag{363}$$

To obtain the equal-index correlator $\Big(\mathcal{M}\Big)^2_{\text{color}(k,k)}$ within this definition, we only need to replace $\delta_{\overline{\omega}_k\omega_k}$ with $\mathcal{T}^c_{\overline{\omega}_k\omega}\mathcal{T}^c_{\omega\omega_k}$.

E Some integrals

In the thesis we encounter the following integrals:

$$I_A = \int d^{2+\overline{\epsilon}}r_T \frac{\theta(0<r_T^2<\infty)}{r_T^2|r_T+k_T|^2} = \frac{\pi_\epsilon |k_T|^{\overline{\epsilon}}}{|k_T|^2}\frac{4}{\overline{\epsilon}} + \mathcal{O}(\overline{\epsilon}^2), \tag{364}$$

$$\begin{aligned} I_{C1} &= \int_0^u \frac{dx}{x}\int d^{2+\overline{\epsilon}}r_T \frac{\theta(0<r_T^2<vx)}{r_T^2|r_T+k_T|^2}, \quad \frac{|k_T|^2}{uv}<1, \\ &= \frac{\pi_\epsilon |k_T|^{\overline{\epsilon}}}{|k_T|^2}\left\{\frac{4}{\overline{\epsilon}^2} - \frac{4}{\overline{\epsilon}}\ln\frac{|k_T|^2}{uv} + \mathrm{Li}_2\Big(\frac{|k_T|^2}{uv}\Big)\right\} + \mathcal{O}(\overline{\epsilon}), \end{aligned} \tag{365}$$

$$\begin{aligned} I_{C2} &= \int_0^u \frac{dx}{x}\int d^{2+\overline{\epsilon}}r_T \frac{\theta(0<r_T^2<vx^2)}{r_T^2|r_T+k_T|^2}, \quad \frac{|k_T|^2}{u^2v}<1, \\ &= \int_0^{u^2} \frac{d\sqrt{x}}{\sqrt{x}}\int d^{2+\overline{\epsilon}}r_T \frac{\theta(0<r_T^2<vx)}{r_T^2|r_T+k_T|^2} = \frac{1}{2}I_{C1}(u\to u^2), \end{aligned} \tag{366}$$

$$
\begin{aligned}
I_D &= \int_0^1 \frac{dx}{x} \int d^{2+\bar{\epsilon}} r_T \frac{1}{|r_T|^2 |r_T + k_T|^2} \theta\left(\frac{|r_T|}{v\sqrt{\Lambda}} < x_r < \frac{|r_T|}{|r_T + k_T|} \right) \\
&= \frac{\pi_\epsilon |k_T|^{\bar{\epsilon}}}{|k_T|^2} \frac{2}{\bar{\epsilon}} \ln \frac{v^2 \Lambda}{|k_T|^2} + \mathcal{O}(\bar{\epsilon}) + \mathcal{O}(\Lambda^{-1}).
\end{aligned}
\tag{367}
$$

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